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On HK_r- and P_r-integrable Functions

Piotr Sworowski

Casimirus the Great University, Institute of Mathematics

Powstańców Wielkopolskich 2, 85-090 Bydgoszcz, Poland

Abstract :

One of central topics in the generalized integration theory in the real line or, more generally, in $\mathbb{R}^n$, is the correspondence between integral defined in the realm of integral sums (e.g., Riemann sums) and antidifferentiation. A standpoint result in this connection, valid in various settings (i.e., for differentiation bases), claims that generalized antidifferentiation (a Perron-type integral) and the integration w.r.t. Riemann sums (a Henstock–Kurzweil-type integral) are equivalent if defined correspondingly to each other. This result, however, need not extend to all possible settings, in particular to those where generalized derivates and integral sums are defined with respect to some integral means.

Musial and Sagher in [2] made an attempt to construct a Henstock–Kurzweil-type integral (under the name *HK_r-integral*) which would cover the so-called P_r-integral [1], a Perron-type integral based on derivates defined via L^r integral means.

Definition 1 *A function $f\colon [a,b] \to \mathbb{R}$ is L^r-Henstock–Kurzweil integrable (HK_r-integrable) on $[a,b]$ if there exists a function $F \in L^r[a,b]$ such that for any $\varepsilon > 0$ there exists a gauge δ such that for any δ-fine partition $\{([c_i,d_i],x_i)\}_{i=1}^q$ in $[a,b]$ we have*

$$\sum_{i=1}^{q} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y-x_i)|^r \, dy \right)^{1/r} < \varepsilon.$$

This attempt wasn't successful in the sense that these two integral turned out to be nonequivalent, a result announced quite recently in [3]. We demonstrate that this nonequivalence is actually much more dramatic and holds for all pairs of parameters $r, s \geq 1$ (HK_r-integral vs. P_s-integral).

Keywords : Kurzweil–Henstock integral; Perron integral; HK_r-integral; P_r-integral.

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References

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