A generalization of the ψ-Hilfer fractional operator

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Abstract: In this presentation: first, we generalize the ψ-Hilfer fractional derivative and set some of the generalized operator's properties, we give a generalized Gronwall inequality with the properties of the functions k-Gamma, k-Beta and k-Mittag-Leffler taken into account, we present the definitions of the k-Mittag-Leffler-Ulam-Hyers stability and some related remarks.
Second, to apply our definitions, we prove some existence, uniqueness and k-Mittag-Leffler-Ulam-Hyers stability results for the boundary valued problem for implicit nonlinear fractional differential equations and k-Generalized ψ-Hilfer fractional derivative:

$$\begin{cases} \left({}^H_k\mathcal{D}^{\vartheta,r;\psi}_{a+}x\right)(t) = f\left(t, x(t), \left({}^H_k\mathcal{D}^{\vartheta,r;\psi}_{a+}x\right)(t)\right), \quad t \in (a,b], \\ c_1\left(\mathcal{J}^{k(1-\xi),k;\psi}_{a+}x\right)(a^+) + c_2\left(\mathcal{J}^{k(1-\xi),k;\psi}_{a+}x\right)(b) = c_3, \end{cases}$$

where ${}^H_k\mathcal{D}^{\vartheta,r;\psi}_{a+}, \mathcal{J}^{k(1-\xi),k;\psi}_{a+}$ are the k-generalize ψ-Hilfer fractional derivative of order $\vartheta \in (0,1)$ and type $r \in [0,1]$, and k-generalize ψ-fractional integral of order $k(1-\xi)$, where $\xi = \frac{1}{k}(r(k-\vartheta)+\vartheta)$, $k > \vartheta$, $f \in C([a,b]\times\mathbb{R}^2, \mathbb{R})$ and $c_1, c_2, c_3 \in \mathbb{R}$ such that $c_1 + c_2 \neq 0$.
Third, we consider the initial value problem with nonlinear implicit k-generalize ψ-Hilfer type fractional differential equation :

$$\begin{cases} \left({}^H_k\mathcal{D}^{\vartheta,r;\psi}_{a+}x\right)(t) = f\left(t, x(t), \left({}^H_k\mathcal{D}^{\vartheta,r;\psi}_{a+}x\right)(t)\right), \quad t \in (a,b], \\ \left(\mathcal{J}^{k(1-\xi),k;\psi}_{a+}x\right)(a^+) = x_0, \end{cases}$$

where $x_0 \in E$ and $f \in C([a,b]\times E\times E, E)$ with $(E, \|\cdot\|)$ a Banach space.
Finally, we consider the boundary valued problem with nonlinear implicit k-generalize ψ-Hilfer type fractional differential equation involving both retarded and advanced arguments:

$$\begin{cases} \left({}^H_k\mathcal{D}^{\vartheta,r;\psi}_{a+}x\right)(t) = f\left(t, x_t(\cdot), \left({}^H_k\mathcal{D}^{\vartheta,r;\psi}_{a+}x\right)(t)\right), \quad t \in (a,b], \\ \alpha_1\left(\mathcal{J}^{k(1-\xi),k;\psi}_{a+}x\right)(a^+) + \alpha_2\left(\mathcal{J}^{k(1-\xi),k;\psi}_{a+}x\right)(b) = \alpha_3, \\ x(t) = \varpi(t), \quad t \in [a-\lambda, a], \ \lambda > 0, \\ x(t) = \tilde{\varpi}(t), \quad t \in \left[b, b+\tilde{\lambda}\right], \ \tilde{\lambda} > 0, \end{cases}$$

where $f : [a,b]\times C\left(\left[-\lambda, \tilde{\lambda}\right], \mathbb{R}\right)\times\mathbb{R} \longrightarrow \mathbb{R}$ is a given appropriate function specified latter and $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that $\alpha_1 + \alpha_2 \neq 0$. For each function x defined on $\left[a-\lambda, b+\tilde{\lambda}\right]$ and for any $t \in (a,b]$, we denote by x_t the element defined by

$$x_t(\tau) = x(t+\tau), \quad \tau \in \left[-\lambda, \tilde{\lambda}\right].$$

Examples are given for justifying our results and addressing the different specific cases of our problems.
Keywords: ψ-Hilfer fractional derivative, Generalized Gronwall Inequality, Mittag-Leffler function, Ulam-Hyers stability.

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