People's Democratic Republic of Algeria Ministry of Higher Education and Scientific Research

Djillali Liabes University of Sidi Bel Abbes

Faculty of Technology

Department of Mechanical Engineering

الجمهورية الجزائرية الديمقراطية الشعبية وزارة
التعليم العالي و البحث العلمي جامعة جيلالي
ليابس - سيدي بلعباس كلية التكنولوجيا
قسم الهندسة الميكانيكية

Educational handout

Rational Mechanics

Course intended for students :

Sector: Civil Engineering, Mechanical Engineering, Public Works, Aeronautics and Hydraulics

Specialization: /

Course: Bachelor's Degree

Semester: S3

Course Unit : UEF 2.1.2

Course presented by:

Dr LIAMANI Samira

Academic year 2024/2025

FOREWORD

This manual is a basic course in Physics 4, titled Rational Mechanics. It is part of the core curriculum for the 3rd semester of the common foundational program in the sciences and engineering field. It is intended for second-year undergraduate students (LMD system). This document adheres to the syllabus of Rational Mechanics for the fields of Civil Engineering, Mechanical Engineering, Public Works, Aeronautics, and Hydraulics, as taught at the Faculty of Technologies of Djillali Liabes University in Sidi Bel Abbès. It is written in the form of detailed lectures with solved applications. The content is organized into five chapters.

The first chapter serves as a mathematical review aimed at providing students with the foundational knowledge required for understanding the course. It includes operations on vectors, moments, and torsors.

The second chapter addresses the statics of rigid bodies. It introduces fundamental concepts in statics, such as material points, perfect rigid bodies, forces, moments, wrenches, the equilibrium of force systems, constraints, reactions, operations on forces, and the equilibrium of solids in the presence of friction.

The third chapter presents the kinematics of rigid bodies, focusing on mechanical motion from a purely geometric perspective, without considering the causes of the motion.

The fourth chapter deals with concepts related to mass, the center of mass, moments of inertia, and products of inertia, highlighting their mechanical significance in the study of kinetics and dynamics.

The fifth and final chapter of this document focuses on the fundamental principle of the dynamics of material systems. The primary objective of this chapter is to study the general theorems governing dynamics.

Dr LIAMANI Samira

Introduction

Rational mechanics is a fundamental branch of classical physics that deals with the laws of motion and the equilibrium of bodies. It relies on rigorous mathematical principles and applies fundamental mechanical concepts such as force, motion, mass, and energy to various physical systems.

Its primary goal is to analyze and predict the behavior of material systems based on the interactions they experience, whether gravitational, elastic, or due to external forces. The discipline traces its origins to the works of great scientists such as Isaac Newton, with his famous laws of motion, and Joseph-Louis Lagrange, who developed an analytical reformulation using the calculus of variations.

Rational mechanics plays a central role in scientific education, as it fosters analytical skills and rigor in modeling physical phenomena. It also serves as a conceptual bridge to more advanced theories such as analytical mechanics and fluid mechanics.

In summary, rational mechanics is a cornerstone of science, enabling the explanation and prediction of natural phenomena with remarkable precision while laying the groundwork for modern scientific and technological advancements.

Table of Contents

Chapter III: Kinematics of the Rigid Body

Chapter IV: Geometry of Masses

Chapter V: Dynamics of the Rigid Body

Chapter I

Mathematical Tools

Chapter I: Mathematical Tools

I.1 Introduction

The modeling of real space, considered within the framework of classical mechanics as being three-dimensional, homogeneous and isotropic, supposes the introduction of mathematical tools such as vectors, and notions about torsos.

In this part we will present the reminders and all the mathematical operations on vectors. We will also develop the study on torsos which are very important mathematical tools in classical mechanics, particularly in solid mechanics. The use of torsors in mechanics makes it possible to simplify the writing of equations relating to the fundamental quantities of mechanics.

I.2 Definition of a vector

Some quantities cannot be described by real or scalar numbers because it is necessary to specify their intensity, their direction and their sense. This pushes us to use vectors to represent them.

We call vector ($\overrightarrow{AB}$) a line segment having an origin (A) and an endpoint (B) and defined by:

His origin (A);

Its direction (the right (Δ));

Its sense (from point A to point B);

Its length or magnitude (the distance AB).

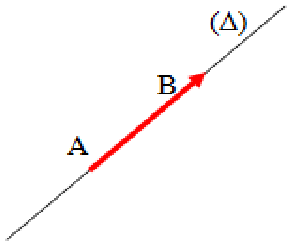
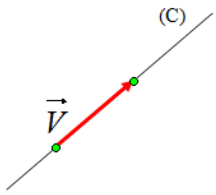


Figure I.1: Graphical presentation of a vector

Vectors are commonly represented by arrows.

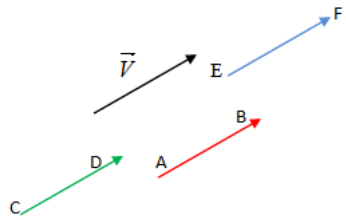
I.2.1 Types of vectors

The vector can be represented in several types:



- A vector is said to be linked if its point of application is fixed (fixed vector).

Example: The position of the vector is completely defined on the support right (C).

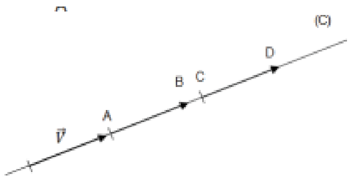


- A vector is said to be free if its point of application and its direction are unknown and its other components are known. (free vector)

Example: The vectors $\overrightarrow{AB}$, $\overrightarrow{CD}$ and $\overrightarrow{EF}$ are representatives of the same vector $\vec{V}$.

- A vector is said to be sliding if its point of application is not fixed. (sliding vector)

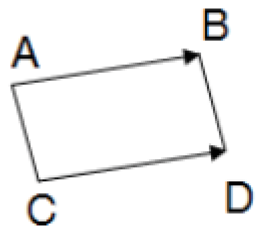
Example: Vectors $\overrightarrow{AB}$, $\overrightarrow{CD}$ are representatives of the same vector $\vec{V}$.



- A vector is said to be unitary if its module is equal to 1.

I.2.2 Vector calculation

I.2.2.1 Equal vectors



Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are considered equal if they have the same length, the same direction and the same sense.

This equality makes ABDC a parallelogram.

I.2.2.2 Addition of vectors

Given two vectors $\overrightarrow{V_1}$ and $\overrightarrow{V_2}$ with $\forall$ $\overrightarrow{V_1}$, $\overrightarrow{V_2}$ $\in$ R3 the sum of these vectors is a vector $\overrightarrow{V_3} \in R^3$

The sum of these two vectors is carried out by transporting the origins of the two vectors to a single point A in order to construct a parallelogram whose sides are $\overrightarrow{V_1}$ and $\overrightarrow{V_2}$. The resulting vector $\overrightarrow{V_3}$ is defined by: $\overrightarrow{V_3} = \overrightarrow{V_1} + \overrightarrow{V_2}$.

If (a_1, a_2, a_3) and (b_1, b_2, b_3) are the components of the vectors $\overrightarrow{V_1}$ and $\overrightarrow{V_2}$ respectively:

$$\overrightarrow{v_1} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k} \; ; \; \overrightarrow{v_2} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$$

The sum of the two vectors:

$$\overrightarrow{V_3} = \overrightarrow{V_1} + \overrightarrow{V_2} = (a_1 + b_1)\vec{i} + (a_2 + b_2)\vec{j} + (a_3 + b_3)\vec{k}$$

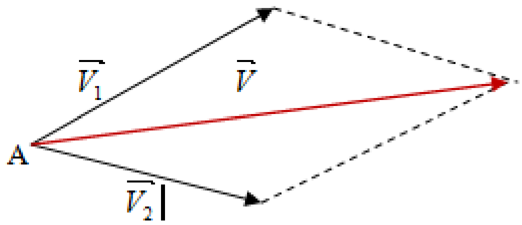


Figure I.2: Parallelogram law

From the construction of the parallelogram, we can deduce another graphical method for the addition of vectors. This method is known as the triangle law. We will only be able to draw half of the parallelogram. In order to add two vectors, we think of them as displacements. We carry out the first displacement, and then the second. So the second displacement must start where the first one finishes.

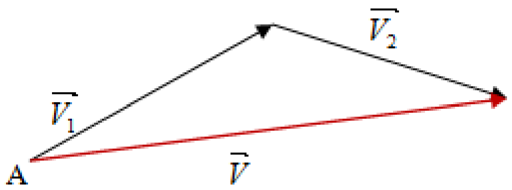


Figure I.3: Triangle law

One of the things we can do with vectors is to add them together. We shall start by adding two vectors together. Once we have done that, we can add any number of vectors together by adding the first two, then adding the result to the third, and so on. This way of proceeding graphically translates the polygon law.

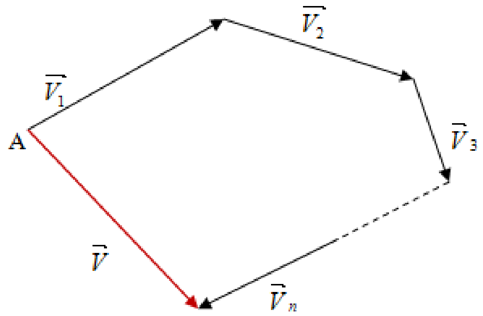


Figure I.4: Polygon law

On the other hand, the summation of the vectors is:

- Commutative: we can add vectors in any order we want say that vector addition is: $\vec{V}_1 + \vec{V}_2 = \vec{V}_2 + \vec{V}_1$
- Associative: $\vec{V}_1 + (\vec{V}_2 + \vec{V}_3) = (\vec{V}_2 + \vec{V}_1) + \vec{V}_3$
- Distributive compared with the vector sum $\lambda(\vec{V}_1 + \vec{V}_2) = \lambda\vec{V}_1 + \lambda\vec{V}_2$
- Distributive compared with the scalar sum: $\vec{V}(\lambda_1 + \lambda_2) = \lambda_1\vec{V} + \lambda_2\vec{V}$
- Identity Element for Vector Addition: There is a unique vector, $\vec{O}$,that acts as an identity element for vector addition. For all vectors $\vec{V}_1$: $\vec{V}_1 + \vec{O} = \vec{V}_1$
- Inverse Element for Vector Addition: For every vector $\vec{V}$, there is a unique inverse vector $\vec{V} \equiv -\vec{V}$ such that $\vec{V} + (-\vec{V}) = \vec{O}$

I.2.2.3 Subtraction of vectors

Subtraction of two vectors $\vec{V}_1 - \vec{V}_2$ is the vector $\vec{V}$ defined as the addition of the vector $\vec{V}_1$ to a vector $\vec{V}_2'$ equal and opposite to $\vec{V}_2$.

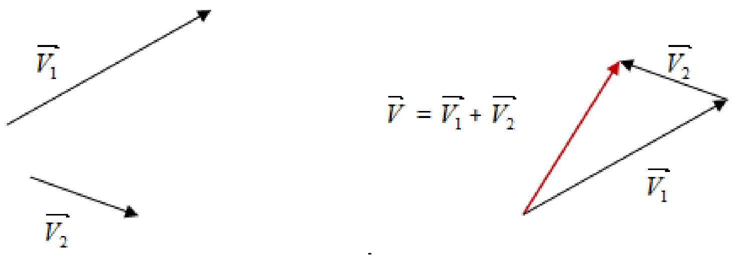


Figure I.5: Subtraction of two vectors

I.2.2.4 Scalar Multiplication of Vectors

Vectors can be multiplied by real positive numbers. Let x be a real number and a $\vec{V_1}$ vector, then the multiplication of $\vec{V_1}$ by x is a new vector $\vec{V_2}$:

If the vector $\vec{V_1}$ has components (a_1, a_2, a_3) such as $\vec{v_1} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

The vector $\vec{V_2}$ would be written: $\vec{v_2} = xa_1\vec{i} + xa_2\vec{j} + xa_3\vec{k}$ z

Scalar multiplication of vectors satisfies the following properties:

a) Associative Law for Scalar Multiplication: $\lambda_1 (\lambda_2\vec{V}) = \lambda_1\lambda_2\vec{V}$.

b) Distributive Law for Vector Addition: $\lambda(\vec{V_1} + \vec{V_2}) = \lambda\vec{V_1} + \lambda\vec{V_2}$;

c) Distributive Law for Scalar Addition-: $(\lambda_1 + \lambda_2)\vec{V} = \lambda_1\vec{V} + \lambda_2\vec{V}$;

d) Identity Element for Scalar Multiplication: The number 1 acts as an identity element for multiplication, $1.\vec{V} = \vec{V}$.

I.2.2.5 Modules of a vector (Standard)

Let the vector $\vec{v} = x\vec{i} + y\vec{j} + z\vec{k}$

The Pythagorean Theorem is used to calculate the modulus or length of the vector, Such as:

$$\left\|\vec{V}\right\| = \sqrt{x^2 + y^2 + z^2}$$

Application example

Let A (2, 5) and B (-3, 1) be two points and $\overrightarrow{AB}$ a vector.

Determine the coordinates and modulus of the vector $\overrightarrow{AB}$.

$$\overrightarrow{AB} = \begin{pmatrix} x_B - x_A \\ y_B - y_A \end{pmatrix} = \begin{pmatrix} -3-2 \\ 1-5 \end{pmatrix}, \ \overrightarrow{AB} = \begin{pmatrix} -5 \\ -4 \end{pmatrix}$$

The vector module: $\left\|\overrightarrow{AB}\right\| = \sqrt{(-5)^2 + (-4)^2}$ so $\left\|\overrightarrow{AB}\right\| = \sqrt{45}$

I.2.3 Decomposition of vectors

We have shown so far that it is always possible to replace two or more vectors by a single vector. Conversely, it is always possible to replace a single vector $\vec{V}$ by two or more vectors. These vectors are called the components of the original vector $\vec{V}$. We must consider two cases of particular interest:

1. One of the components $\overrightarrow{V_1}$ is fixed. We calculate the second component using the triangle law.

2. The two directions of decomposition are given. The magnitude and orientation of the components are obtained by applying the parallelogram principle.

I.2.4The vectors Product

There are two kinds of multiplication involving vectors. The first is known as the scalar product. The second product is known as the vector product.

I.2.4.1 Scalar product of two vectors

Let there be two vectors $\overrightarrow{V_1}$ and $\overrightarrow{V_2}$ their scalar product is a product which gives as result a scalar, (Fig. I.6): $\vec{V_1}.\vec{V_2} = \left\|\vec{V_1}\right\|.\left\|\vec{V_2}\right\| \cos\theta$

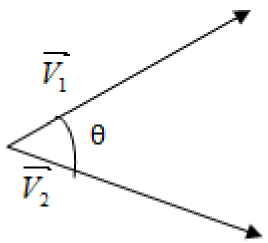


Figure I.6: Scalar product

Such that θ is the angle between the two vectors. The angle θ is always chosen to lie between 0 and π, and the tails of the two vectors must coincide.

The scalar product of two vectors is:

- Commutative: $\overrightarrow{V_1} \cdot \overrightarrow{V_2} = \overrightarrow{V_2} \cdot \overrightarrow{V_1}$;
- Associative with respect to the multiplication of a scalar: $\lambda(\overrightarrow{V_1} \cdot \overrightarrow{V_2}) = \overrightarrow{V_1} \cdot (\lambda \cdot \overrightarrow{V_2})$;

- Distributive compared with vector sum: $\vec{V_1}.(\vec{V_2}+\vec{V_3})=\vec{V_1}.\vec{V_2}+\vec{V_1}.\vec{V_3}$;

- Generally, whenever any two vectors are perpendicular to each other their scalar product is zero because the angle between the vectors is 90° and cos90° = 0: $\vec{V_1}\perp\vec{V_2}\Rightarrow\vec{V_1}.\vec{V_2}=0$

Analytical expression

The scalar product can be defined by the analytical expression: $\vec{V_1}.\vec{V_2}=a_1b_1+a_2b_2+a_3b_3$

Example

$\vec{V_1}=3\vec{x}+2\vec{y}-1\vec{z}$; $\vec{V_2}=4\vec{x}+1\vec{y}+7\vec{z}$

$\vec{V_1}.\vec{V_2}=(3\times 4)+(2\times 1)+(-1\times 7)$; $\vec{V_1}.\vec{V_2}=7$

I.2.4.2 Vector product of two vectors

The vector product is a vector operation carried out in oriented Euclidean spaces of dimension 3, this operation does not exist in 2 dimensions.

Consider two vectors $\vec{V_1}$ and $\vec{V_2}$ with $\forall\ \vec{V_2}, \vec{V_1} \in R^3$

The vector product of these two vectors is a vector $\vec{V_3} \in R^3$ such that: $\vec{V_3}$ is a vector perpendicular to $\vec{V_1}$ and $\vec{V_2}$

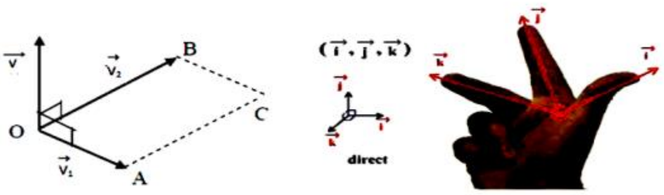


Figure I.7: Vector product

$$\vec{V_1}\wedge\vec{V_2}=\vec{V_3}=\|\vec{V_1}\|.\|\vec{V_2}\|\sin\alpha.\vec{n}$$

$\vec{n}$ is a unit vector perpendicular to the plane containing a by $\vec{V_1}$ and $\vec{V_2}$ in the sense defined by the right-handed screw rule;

• The modulus of the vector product is equal to the area of the parallelogram formed by $\vec{V_1}$ and $\vec{V_2}$;

• The vector product is distributive on the left and right for the vector sum:

$$\left(\vec{V}_1 + \vec{V}_2\right) \wedge \vec{V}_3 = \vec{V}_1 \wedge \vec{V}_3 + \vec{V}_2 \wedge \vec{V}_3$$

$$\vec{V}_3 \wedge \left(\vec{V}_1 + \vec{V}_2\right) = \vec{V}_3 \wedge \vec{V}_1 + \vec{V}_3 \wedge \vec{V}_2$$

• The vector product is associative for multiplication by a real number:

$$\lambda.\vec{V}_1 \wedge \vec{V}_2 = \lambda (\vec{V}_1 \wedge \vec{V}_2)$$

$$\vec{V}_1 \wedge \lambda \vec{V}_2 = \lambda (\vec{V}_1 \wedge \vec{V}_2)$$

• The vector product is <u>antisymmetric (anticommutative)</u>

$$\vec{V}_1 \wedge \vec{V}_2 = - \vec{V}_2 \wedge \vec{V}_1$$

If we apply this property to the vector product of the same vector, we will have:

$$\vec{V}_1 \wedge \vec{V}_1 = -(\vec{V}_1 \wedge \vec{V}_1) = \vec{O}$$

• We deduce from this property that the vector product is null if:

- The two vectors are collinear;
- One of the vectors is null.

$$\vec{V}_1 \; // \; \vec{V}_1 \text{ so } \vec{V}_1 \wedge \vec{V}_1 = \vec{O}$$

Analytical Expression

The vector product can be calculated by the direct method in Cartesian coordinates in a direct orthonormal coordinate system:

$$\vec{V}_1 = \alpha\vec{x} + \beta\vec{y} + \gamma\vec{z}$$

$$\vec{V}_2 = \delta\vec{x} + \sigma\vec{y} + \tau\vec{z}$$

$$\vec{V}_1 \wedge \vec{V}_2 = (\alpha\vec{x} + \beta\vec{y} + \gamma\vec{z}) \wedge (\delta\vec{x} + \sigma\vec{y} + \tau\vec{z})$$

$$\vec{V}_1 \wedge \vec{V}_2 = (\beta\tau - \gamma\sigma)\vec{x} - (\alpha\tau - \gamma\delta)\vec{y} + (\alpha\sigma - \beta\delta)\vec{z}$$

It can also be determined by the determinant method:

$$\overrightarrow{V_1} \wedge \overrightarrow{V_2} = \begin{vmatrix} \vec{x} & \vec{y} & \vec{z} \\ \alpha & \beta & \gamma \\ \delta & \sigma & \tau \end{vmatrix} = \begin{vmatrix} \beta & \gamma \\ \sigma & \tau \end{vmatrix} \vec{x} - \begin{vmatrix} \alpha & \gamma \\ \delta & \tau \end{vmatrix} \vec{y} + \begin{vmatrix} \alpha & \beta \\ \delta & \sigma \end{vmatrix} \vec{z}$$

$$\overrightarrow{V_1} \wedge \overrightarrow{V_2} = (\beta\tau - \gamma\sigma)\vec{x} - (\alpha\tau - \gamma\delta)\vec{y} + (\alpha\sigma - \beta\delta)\vec{z}$$

Application example

$$\overrightarrow{V_1} = 3\vec{x} + 2\vec{y} - 1\vec{z}$$

$$\overrightarrow{V_2} = 4\vec{x} + 1\vec{y} + 7\vec{z}$$

$$\overrightarrow{V_1} \wedge \overrightarrow{V_2} = \begin{vmatrix} \vec{x} & \vec{y} & \vec{z} \\ 3 & 2 & -1 \\ 4 & 1 & 7 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 1 & 7 \end{vmatrix} \vec{x} - \begin{vmatrix} 3 & -1 \\ 4 & 7 \end{vmatrix} \vec{y} + \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} \vec{z}$$

$$\overrightarrow{V_1} \wedge \overrightarrow{V_2} = (14+1)\vec{x} - (21+4)\vec{y} + (3-8)\vec{z}$$

$$\overrightarrow{V_1} \wedge \overrightarrow{V_2} = 15\vec{x} - 25\vec{y} - 5\vec{z}$$

I.2.4.3 The mixed product

We call the mixed product of three vectors $\overrightarrow{V_1}, \overrightarrow{V_2} and \overrightarrow{V_3}$ taken in this order, the real number defined by: $\overrightarrow{V_1}.(\overrightarrow{V_2} \wedge \overrightarrow{V_3})$

The absolute value of the mixed product is the volume of the parallelepiped formed by the 3 vectors.

The mixed product is null if:

- The three vectors are in the same plane;
- Two of the vectors are collinear;
- One of the vectors is null.

It is easily shown that, in a direct orthonormal basis, the mixed product is a scalar variant by direct circular permutation of the three vectors because the scalar product is commutative:

$$\overrightarrow{V_1}.(\overrightarrow{V_2} \wedge \overrightarrow{V_3}) = \overrightarrow{V_3}.(\overrightarrow{V_1} \wedge \overrightarrow{V_2}) = \overrightarrow{V_2}.(\overrightarrow{V_1} \wedge \overrightarrow{V_3})$$

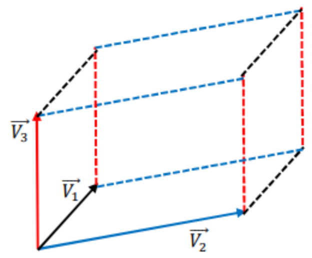


Figure I.8: Mixed product of three vectors

I.2.4.4 The sine rule in a triangle

Study the triangle ABC, we can establish a relationship between the three sides and the three angles of the triangle.

In the triangles ABD and CBD, we have:

$\sin\alpha = \dfrac{DB}{AB}$, and $\sin\beta = \dfrac{DB}{BC}$ from where

AB sinα = BC sin β, we deduce: $\dfrac{BC}{\sin\alpha} = \dfrac{AB}{\sin\beta}$

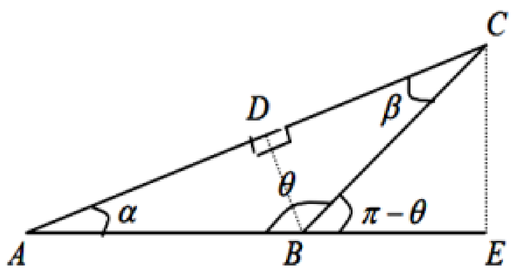


Figure I.9: Rule of sinus in a triangle

Likewise for the triangles AEC and BEC:

We have: $\sin\alpha = \dfrac{EC}{AC}$, and $\sin(\pi-\theta) = \dfrac{EC}{BC}$

From where AC sinα = BC sin (π − θ) = BC sin θ

We deduce: $\dfrac{BC}{\sin\alpha} = \dfrac{AC}{\sin\theta}$

We finally deduce a relationship called the sine rule in a triangle:

$$\frac{BC}{\sin\alpha} = \frac{AB}{\sin\beta} = \frac{AC}{\sin\theta}$$

1.2.5 Projection of vectors

I.2.5.1 Orthogonal projection of a vector on an axis

Let be any vector, and an axis (Δ) defined by its unit vector $\vec{u}$.

The orthogonal projection of the vector $\vec{V}$ on the axis (Δ) defined by the component $\overrightarrow{V_x}$ of this vector on this axis.

$\overrightarrow{V_x} = (\vec{V}.\vec{u})\vec{u}$

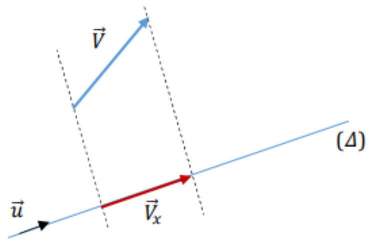


Figure I.10: Orthogonal projection of a vector onto an axis

I.2.5.2 Orthogonal projection of a vector onto a plane

Given $\vec{V}$, an arbitrary vector, its projection onto the plane (π) defined by the normal $\vec{n}$ is the component $\overrightarrow{V_\pi}$ in the plane.

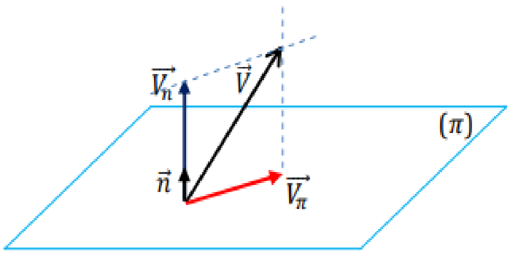


Figure I.11: Orthogonal projection of a vector onto an axis

We can write the projection of $\vec{V}$ on the plane by the following relation: $\overrightarrow{V_\pi} = \vec{V} - \overrightarrow{V_{\vec{n}}}$

Whence $\vec{V} = \vec{V}(\vec{n}.\vec{n})$ And $\overrightarrow{V_{\vec{n}}} = (\vec{V}.\vec{n})\vec{n}$

So: $\overrightarrow{V_\pi} = \vec{V}(\vec{n}.\vec{n}) - (\vec{V}.\vec{n})\vec{n}$

And we find the vector expression of the vector $\overrightarrow{V_\pi}$ by the following double vector relation:

$\overrightarrow{V_\pi} = \vec{n} \wedge (\vec{V} \wedge \vec{n})$

I.2.6 Operators and vectors

I.2.6.1 Gradient operator in an orthonormal frame R (O, $\vec{i},\vec{j},\vec{k}$)

We define the vector operator noted: $\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$ as being the derivative in space along the three directions of the unit vectors.

The gradient of a scalar U is defined as being the vector derivative following the three respective directions $\vec{i},\vec{j},\vec{k}$ with respect to the variables: x, y, z.

$$\overrightarrow{gradU}(x,y,z) = \frac{\partial U}{\partial x}\vec{i} + \frac{\partial U}{\partial y}\vec{j} + \frac{\partial U}{\partial z}\vec{k} \text{ Or } \overrightarrow{gradU} = \vec{\nabla}U$$

Example

U= 3xy-2zx+5yz

$\frac{\partial U}{\partial x}$ =3y-2z, $\frac{\partial U}{\partial y}$ =3x+5z, $\frac{\partial U}{\partial z}$ =-2x+5y

$$\overrightarrow{gradU}(x,y,z) = (3y-2z)\vec{i} + (3x+5z)\vec{j} + (-2x+5y)\vec{k}$$

The gradient of a scalar is a vector.

I.2.6.2 Divergence operator in an orthonormal reference frame R (O, $\vec{i},\vec{j},\vec{k}$)

The divergence of a vector $\vec{V}$ is defined as the scalar product of the operator:

$\vec{V} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$ by the vector ; $\overrightarrow{rot}\vec{V} = \vec{\nabla} \wedge \vec{V}$

$$\overrightarrow{rot}\vec{V} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \wedge \left(V_x\vec{i} + V_y\vec{j} + V_z\vec{k}\right)$$

The rotational of a vector is also a vector.

In matrix form we will have: $\overrightarrow{rot}(\vec{V}) = \begin{cases} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{cases} \wedge \begin{cases} V_x \\ V_y \\ V_z \end{cases} = \begin{cases} \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \\ \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \\ \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \end{cases}$

I.3The Moments (Torque)

A moment represents a vector physical quantity reflecting the ability of this force to turn a mechanical system.

Moments are vectors, and like any vector, they are defined by four parameters that define all vectors: sense, direction, intensity and point of application.

The direction of the moment is determined in accordance with the trigonometric direction (also called geometric direction).

• The positive trigonometric direction corresponds counterclockwise;

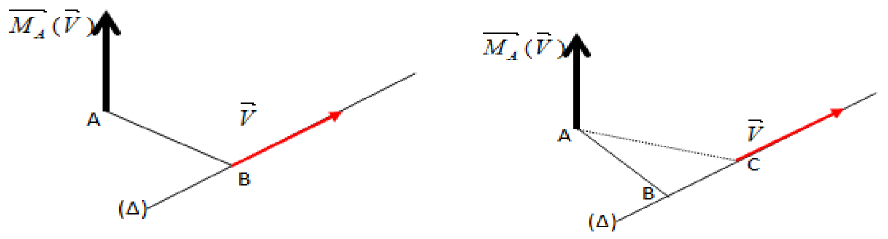
• The negative trigonometric direction corresponds clockwise.

I.3.1 Moment of a vector relative to a point

The moment $\overrightarrow{M_A}$ of a vector $\vec{V}$ of origin B (sliding or fixed) with respect to a point A is equal to the cross product of the position vector $\overrightarrow{AB}$ with the vector $\vec{V}$ (Fig. I.12 (a))

It is written: $\overrightarrow{M_A}(\vec{V}) = \overrightarrow{AB} \wedge \vec{V}$

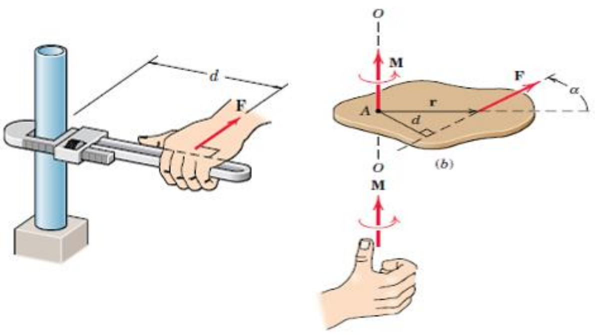
The triad formed respectively by the vectors ($\overrightarrow{AB}, \vec{V}, \overrightarrow{M_A}$) is direct.



(a) **(b)**

Figure I.12: Moment of a vector with respect to a point

Example



Noticed:

The moment at point A is independent of the position of the vector $\vec{V}$ on the axis (Δ). Indeed, we have (Fig. I.12 (b)):

$$\overrightarrow{M_A}(\vec{V}) = \overrightarrow{AC} \wedge \vec{V} = (\overrightarrow{AB} + \overrightarrow{BC}) \wedge \vec{V}$$

But we have: $\overrightarrow{BC} \;//\; \vec{V} \Rightarrow \overrightarrow{BC} \wedge \vec{V} = \vec{0}$ so $\overrightarrow{M_A}(\vec{V}) = \overrightarrow{AB} \wedge \vec{V}$

The moment $\overrightarrow{M_A}(\vec{V})$ is perpendicular to the plane formed by the vectors $\overrightarrow{AB}$ and $\vec{V}$.

Distance AB is often called to as the lever arm.

I.3.2 Moment of a vector with respect to an axis

The moment $\overrightarrow{M_\Delta}(\vec{V})$ of a vector $\vec{V}$ with respect to an axis (Δ) defined by a point A and a unit vector $\vec{u}$, is equal to the projection of the moment $\overrightarrow{M_A}(\vec{V})$ on the axis (Δ).

$$\overrightarrow{M_\Delta}(\vec{V}) = (\overrightarrow{M_A}(\vec{V}).\vec{u})\vec{u}$$

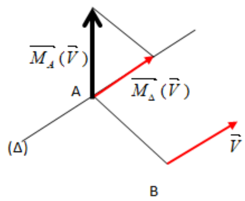
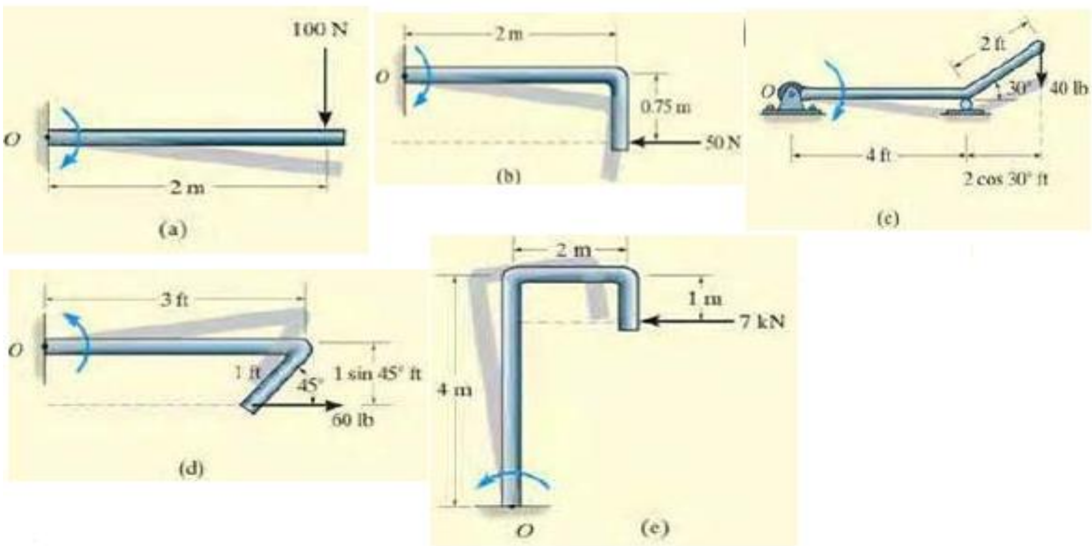


Figure I.13: Moment of a vector with respect to an axis

Example

For each case illustrated in the figure, determine the moment of the force about point O



Solution:

Fig (a) M_O =(100 N)(2m) = 200 N.m   Fig (b) M_O = (50 N)(0.75 m) = 37.5 N.m

Fig (c) M_O =(40 lb)(4 ft+2cos30 ft) = 299 lb.ft

Fig (d) M_O =(60 lb)(1sin45 ft) = 42.4 lb.ft   Fig (e) M_O =(7 kN)(4m−1m)=21 kN.m

I.4The Torsors

Tensors are mathematical tools widely used in mechanics. The use of tensors in the study of complex mechanical systems is very convenient because it simplifies the writing of vector equations. A vector equation represents three scalar equations, and a tensor equation is equivalent to two vector equations, thus to six scalar equations. There are four different types of tensors: the kinematic tensor, the kinetic tensor, the dynamic tensor, and the action tensor.

I.4.1 Definition of a Torsor

A tensor, which we will denote as [T], is defined as a set of two vector fields defined in the geometric space and having the following properties:

a) The first vector field associates with every point A in space a vector independent of point A, called the resultant of the tensor [T].

b) The second vector field associates with every point A in space a vector that depends on point A. This vector is called the moment at point A of the tensor [T].

I.4.2 Rating

The resultant $\vec{R}$ and the resultant moment $\overrightarrow{M_A}$ at point A constitute the reduction elements of the torso at point A.

Let $\vec{R}$ be the resultant of the n sliding vectors: $\vec{V_1},\vec{V_2},\vec{V_3}..........\vec{V_n}$ applied respectively to the points: B_1, B_2, B_3……B_n. We can define two quantities from this system of vectors:

- The resultant of the n vectors: $\vec{R}=\sum_{i=1}^{n}\overrightarrow{V_i}$;
- The resulting moment at a point A in space is given by: $\overrightarrow{M_A}=\sum_{i=1}^{n}\overrightarrow{AB_i}\wedge\overrightarrow{V_i}$.

The two quantities constitute the torsor developed at point A associated with the given vector system. We adopt the following notation: $[T]_A=\begin{cases}\vec{R}\\ \overrightarrow{M_A}\end{cases}$

Note: A torsor is not equal to a vector pair, but it is represented at point A by its reduction elements.

I.4.3 Operation on the torsors

Be the torsos [T_1] and [T_2] such as $[T_1]=\begin{cases}\overrightarrow{R_1}\\ \overrightarrow{M_1}\end{cases}$ and $[T_2]=\begin{cases}\overrightarrow{R_2}\\ \overrightarrow{M_2}\end{cases}$ and λ a scalar

I.4.3.1 Sum of two torsors

The sum of two torsors [T1] and [T2] is a torsor [T] whose reduction elements $\vec{R}$ and $\overrightarrow{M_A}$ are respectively the sum of the reduction elements of the two torsors.

$$[T]_A = [T_1]_A + [T_2]_A \Leftrightarrow [T]_A = \begin{cases}\vec{R} = \overrightarrow{R_1} + \overrightarrow{R_2}\\ \vec{M} = \overrightarrow{M_1} + \overrightarrow{M_2}\end{cases}$$

I.4.3.2 Multiplication of a torsor by a scalar

The multiplication of a torsor by a scalar is given by the following expression:

$$\lambda[T_1] = \begin{cases}\lambda.\overrightarrow{R_1}\\ \lambda.\overrightarrow{M_1}\end{cases} \text{ with } \lambda \in R$$

I.4.3.3 Equality of two torsors

Two torsors are equal (equivalent), if and only if there exists a point in space at which the reduction elements are respectively equal to each other. Let two torsors [T_1] and [T_2] be such that: [T_1]A = [T_2]A equal to point A, this equality results in two vector equalities:

$$[T_1]_A = [T_2]_A \Leftrightarrow \begin{cases}\overrightarrow{R_1} = \overrightarrow{R_2}\\ \overrightarrow{M_{1A}} = \overrightarrow{M_{2A}}\end{cases}$$

I.4.3.4 Product of two torsors

We call the product of the two torsors $[T]_1$ and $[T]_2$ the real defined by:

$$\phi A = [T]_1 \otimes [T]_2 = \overrightarrow{R_1}.\overrightarrow{M_{2A}} + \overrightarrow{R_2}.\overrightarrow{M_{1A}}$$

I.4.3.5 Null torsors

The zero torsors denoted [0] is the neutral element for the addition of two torsors. Its reduction elements are zero at any point in space.

$$[0] \Leftrightarrow \begin{cases}\vec{R} = 0\\ \overrightarrow{M_A} = 0\end{cases} \quad \forall A \in R^3$$

I.5 Properties of moment vectors

I.5.1 Moments transport formula

Knowing the Torsor $[T]_A = \begin{cases} \vec{R} = \sum_i \vec{V_i} \\ \overrightarrow{M_A} = \sum_i \overrightarrow{AB_i} \wedge \vec{V_i} \end{cases}$ at a point A in space we can determine the reduction elements of this same torsor at another point C in space.

The moment at point C is expressed as a function of the moment at point A, the resultant $\vec{R}$ and the vector $\overrightarrow{CA}$. We have in fact:

$$\overrightarrow{M_C} = \sum_{i=1}^{n} \overrightarrow{CB_i} \wedge \vec{V_i} = \sum_{i=1}^{n} (\overrightarrow{CA_i} + \overrightarrow{AB_i}) \wedge \vec{V_i} = \sum_{i=1}^{n} \overrightarrow{CA_i} \wedge \vec{V_i} + \sum_{i=1}^{n} \overrightarrow{AB_i} \wedge \vec{V_i} = \overrightarrow{CA_i} \wedge \sum_{i=1}^{n} \vec{V_i} + \sum_{i=1}^{n} \overrightarrow{AB_i} \wedge \vec{V_i}$$

$$\overrightarrow{M_C} = \overrightarrow{CA} \wedge \vec{R} + \overrightarrow{M_A} \quad \text{So} \quad \overrightarrow{M_C} = \overrightarrow{M_A} + \overrightarrow{CA} \wedge \vec{R}$$

This very important relationship in mechanics makes it possible to determine the moment at a point C by knowing the moment at point A.

I.5.2 Equiprojectivity of moment vectors

The moment vectors $\overrightarrow{M_A}$ at point A and $\overrightarrow{M_C}$ at point C have the same projection on the line AC:

We say that the field of moment vectors is equiprojective.

$$\overrightarrow{M_C} = \overrightarrow{M_A} + \overrightarrow{CA} \wedge \vec{R}$$

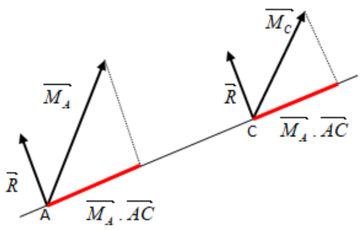


Figure I.14: Equiprojectivity of moment vectors

Projecting the moment vector onto the CA axis amounts to making the scalar product with the vector $\overrightarrow{CA}$ up to a multiplicative factor. We have the transport formula:

$$\overrightarrow{M_C} = \overrightarrow{M_A} + \overrightarrow{CA} \wedge \vec{R}$$

Let us multiply this relation scalarly by the vector $\overrightarrow{CA}$

$$\overrightarrow{CA}.\overrightarrow{M_C} = \overrightarrow{CA}(\overrightarrow{M_A} + \overrightarrow{CA} \wedge \vec{R}) = \overrightarrow{CA}.\overrightarrow{M_A} + \overrightarrow{CA}.(\overrightarrow{CA} \wedge \vec{R})$$

Or $\overrightarrow{CA} \wedge \vec{R}$ is a vector perpendicular to $\overrightarrow{CA}$ then: $\overrightarrow{CA}.(\overrightarrow{CA} \wedge \vec{R}) = 0$

We finally obtain:

$\overrightarrow{CA}.\overrightarrow{M_C} = \overrightarrow{CA}.\overrightarrow{M_A}$ or $\overrightarrow{M_C}.\overrightarrow{CA} = \overrightarrow{M_A}.\overrightarrow{CA}$. The scalar product is commutative.

This expression expresses only the projections of the moment vectors $\overrightarrow{M_C}$ and $\overrightarrow{M_A}$ on the right CA are equal.

I.6 Type of Torsors

I.6.1Torsor Couple

We call a couple, a torso whose result is zero. The moment of a couple is a torso invariant and therefore the scalar and vector invariants are null too.

$$[T]_A = \begin{cases} \vec{R} = 0 \\ \overrightarrow{M_A} \neq 0 \end{cases}$$

Properties of the moment vector

The moment of a torsor couple is independent of the points of the space where it is measured.

We have: $V_1 = V_2$ such as: $\vec{R} = \vec{V_1} + \vec{V_2} = \vec{0} \Rightarrow \vec{V_2} = -\vec{V_1}$

The moment at any point A of the space is given by:

$$\overrightarrow{M_A} = \overrightarrow{AP} \wedge \vec{V_1} + \overrightarrow{AQ} \wedge \vec{V_2} = \overrightarrow{AP} \wedge \vec{V_1} - \overrightarrow{AQ} \wedge \vec{V_1}$$

$$\overrightarrow{M_A} = \overrightarrow{AP} \wedge \vec{V_1} - \overrightarrow{AQ} \wedge \vec{V_1} = \overrightarrow{QP} \wedge \vec{V_1}$$

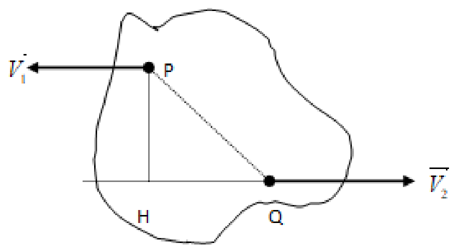


Figure I.15: Torsor Couple

It is clear that the moment at point A is independent of A. We will show that it is also independent of points P and Q.

Indeed we have: $\overrightarrow{M_A} = \overrightarrow{QP} \wedge \overrightarrow{V_1} = (\overrightarrow{QH} + \overrightarrow{HP}) \wedge \overrightarrow{V_1} = \overrightarrow{HP} \wedge \overrightarrow{V_1}$

H is the orthogonal projection of the point P on the right support of the vector $\overrightarrow{V_2}$.

In reality the moment of a torsor couple depends only on the distance that separates the two supporting lines from the two vectors, it is independent of the place where it is measured.

Decomposition of a torsor couple

Let [T] a torso couple defined by: $[T]_C = \begin{cases} \vec{0} \\ \overrightarrow{M_A} \end{cases}$. This torso couple can be broken down into two sliders $[T]_1$ and $[T]_2$ such that: $[T]_C$ = $[T_1]$ +$[T_2]$ where both sliders are defined as follows: $[T]_A = \begin{cases} \overrightarrow{R_1} + \overrightarrow{R_2} = \vec{0} \\ \overrightarrow{M} = \overrightarrow{M_{1P}} + \overrightarrow{M_{2P}} \end{cases}$ where P is any point

The invariants of the two sliders are null: I1= $\overrightarrow{M_{1P}} . \overrightarrow{R_1} = 0$; I2= $\overrightarrow{M_{2P}} . \overrightarrow{R_2} = 0$

There is an infinite solution equivalent to a torso couple.

The problem is solved as follows:

a) Select a slider $[T_1]$ by:

- The result of the slider: $\overrightarrow{R_1}$;

- The axis (Δ1) of the slider, defined by a point P1 such as: (Δ_1)= (P_1, $\overrightarrow{R_1}$)

b) The slider $[T_2]$ is then defined as:

- Its resultant $\overrightarrow{R_2}$ =- $\overrightarrow{R_1}$;

- Its axis (Δ2) is easily determined because it is parallel to (Δ1); it is then enough to know a point P2 of this axis. Point P is determined by the following relationship: $\vec{R}_1 \wedge \overrightarrow{P_1P_2} = \overrightarrow{M}$

This relationship uniquely determines the position of the point.

I.6.2 Sliding Torsor

We call slider any torso of non-zero result which admits a point P for which its moment is null.

This definition can be translated as:[T] is a slider $\Leftrightarrow \begin{cases} \overrightarrow{M_P}.\vec{R} = 0 \\ \qquad\qquad\qquad \forall \text{ P} \\ \vec{R} \neq 0 \end{cases}$

We know that the scalar invariant is independent of the point P where it is calculated. As the result is not null then we can say that: a torso is a slider, if and only if, there is at least one point in which the moment of the torso is zero.

Moment at a point of a slider

Let [T] be a given slider. There is at least one point where the slider moment is zero.

At this point we can write: $\overrightarrow{M_A} = \vec{0}$,

By the transport formula the moment at any point P is written:

$$\overrightarrow{M_P} = \overrightarrow{M_A} + \vec{R} \wedge \overrightarrow{AP}$$

$$\overrightarrow{M_P} = \vec{R} \wedge \overrightarrow{AP}$$

This relation expresses the moment vector at any point P of a slider whose moment is zero at point A.

Axe of a slider

Let [T] be a given slider and A any point such as: $\overrightarrow{M_A} = \vec{0}$,

Let's look at all the P points for which the torso moment is zero:

If $\overrightarrow{M_P} = \vec{0} \Leftrightarrow \vec{R} \wedge \overrightarrow{AP} = \vec{0}$;

This relationship shows that vector $\overrightarrow{AP}$ is collinear to the resultant $\vec{R}$.

The set of points P is determined by the line passing through the point A and unit vector parallel to the resultant $\vec{R}$.

This line is called the zero moment axis of the slider or slider axis. It represents the center axis of the slider.

A non-zero resultant torsor is a slider, if and only if, its scalar invariant is null.

I.6.3 Any torso

A torsor is any, if and only if, its scalar invariant is not null.

[T] is any torso$\Leftrightarrow$ $\overrightarrow{M_P}.\vec{R} \neq \vec{0}$

Decomposition of any torso

Any torsor [T] can be decomposed infinitely into the sum of a sliding torsor [T_1] and a couple torsor [T_2].

We proceed as follows:

a) Select point P

Choose a point P where the torso reduction elements [T] are known: $[T]_A = \begin{cases} \vec{R} \\ \overrightarrow{M_P} \end{cases}$

The choice of point P will depend on the problem to be solved; we choose the easiest point to determine. Once the choice is made, the decomposition of any torso is unique.

b) Slider Construction [T1]

- The result equal to the result of any torso: $\overrightarrow{R_1} = \vec{R}$, with its axis passing through the point P already chosen;

- The moment is zero on this axis: $\overrightarrow{M_{1P}} = \vec{0}$

The slider [T] will have for reduction elements: $[T]_1 = \begin{cases} \overrightarrow{R_1} = \vec{R} \\ \overrightarrow{M_{1P}} = \vec{0} \end{cases}$

c) Couple Torso Construction [T2]

- The result is zero: $\overrightarrow{R_2} = \vec{0}$,

- Couple torso moment is equal to any torso moment: $\overrightarrow{M_{2P}} = \overrightarrow{M_P}$

The slider [T] will have for reduction elements: $[T]_2 = \begin{cases} \overrightarrow{R_2} = \vec{0} \\ \overrightarrow{M_{2P}} = \overrightarrow{M_P} \end{cases}$

We thus obtain [T] = [T_1]+[T_2]

At each point initially chosen we can make this construction. All sliders obtained will have the same result. They differ in their axes but keep the same direction because they are all parallel to the axis carrying the resultant of any torso.

Application exercises

Exercise 01:

Two points A and B, have for Cartesian coordinates in space: A (2, 3, -3), B (5, 7, 2) Determine the components of the vector $\overrightarrow{AB}$ as well as its module, its direction and its direction.

Corrected 01:

Vector $\overrightarrow{AB}$ is given by: $\overrightarrow{AB} = \overrightarrow{OB} + \overrightarrow{OA} = 3\vec{i} + 4\vec{j} + 5\vec{k}$

His module: AB = $\sqrt{3^2 + 4^2 + 5^2} = \sqrt{50}$

His direction: is determined by the angles (α, β, θ) it makes with each of the reference axes.

These angles are deduced by the scalar product of vector $\overrightarrow{AB}$ by the unit vectors of the orthonormed reference frame:

$\alpha = (\overrightarrow{AB}, \vec{i})$: $\overrightarrow{AB}.\vec{i}$ = AB.1 cos α ⇔ cos α = $\frac{\overrightarrow{AB}.\vec{i}}{AB.1} = \frac{3}{\sqrt{50}}$ = 0.424 ⇒ α = 64.89°

$\beta = (\overrightarrow{AB}, \vec{j})$: $\overrightarrow{AB}.\vec{j}$ = AB.1 cos β ⇔ cos β = $\frac{\overrightarrow{AB}.\vec{j}}{AB.1} = \frac{4}{\sqrt{50}}$ = 0.565 ⇒ β = 55.54°

$\theta = (\overrightarrow{AB}, \vec{k})$: $\overrightarrow{AB}.\vec{k}$ = AB.1 cos θ ⇔ cos θ = $\frac{\overrightarrow{AB}.\vec{k}}{AB.1} = \frac{5}{\sqrt{50}}$ = 0.707 ⇒ θ = 44.99°

His sense: as the scalar product of vector $\overrightarrow{AB}$ with the three unit vectors is positive then, it has a positive sense following the three axes of the mark.

Exercise 02:

Be the vectors $\overrightarrow{V_1}$, $\overrightarrow{V_2}$, $\overrightarrow{V_3}$ and $\overrightarrow{V_4}$ such as:

$\overrightarrow{V_1} = \vec{i} + 4\vec{k}$, $\overrightarrow{V_2} = 2\vec{i} + y\vec{j} + z\vec{k}$, $\overrightarrow{V_3} = \vec{i} - 2\vec{j} + 4\vec{k}$, $\overrightarrow{V_4} = 4\vec{i} + y\vec{j} + 2\vec{k}$

1) Determine y and z for vectors $\overrightarrow{V_1}$ and $\overrightarrow{V_2}$ to be collinear,

2) Determine y for vectors $\overrightarrow{V_3}$ and $\overrightarrow{V_4}$ to be perpendicular,

Corrected 02:

1. $\overrightarrow{V_1}$ and $\overrightarrow{V_2}$ are collinear: $\vec{0} = \overrightarrow{V_2} \wedge \overrightarrow{V_1}$ □

$$\overrightarrow{V_1} \wedge \overrightarrow{V_2} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} \wedge \begin{pmatrix} 2 \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -4y \\ 8-z \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} y = 0 \\ z = 8 \end{cases}$$

2. $\overrightarrow{V_3}$ and $\overrightarrow{V_4}$ are perpendicular: $\overrightarrow{V_3} \cdot \overrightarrow{V_4} = 0$

$\overrightarrow{V_3} \cdot \overrightarrow{V_4} = 4 - 2y + 8 = 0 \Leftrightarrow y = 6$

Exercise 03:

Let two torsors $[T_1]_A$ and $[T_2]_A$ be defined at the same point A by their reduction elements in an orthonormal coordinate system: R (O, $\vec{i}, \vec{j}, \vec{k}$) :

$$[T_1]_A = \begin{cases} \overrightarrow{R_1} = -3\vec{i} + 2\vec{j} + 2\vec{k} \\ \overrightarrow{M_{1A}} = 4\vec{i} - \vec{j} - 7\vec{k} \end{cases} \quad \text{and} \quad [T_2]_A = \begin{cases} \overrightarrow{R_2} = 3\vec{i} - 2\vec{j} - 2\vec{k} \\ \overrightarrow{M_{2A}} = 4\vec{i} + \vec{j} + 7\vec{k} \end{cases}$$

1) Determine the central axis and the pitch of the torso $[T_1]_A$;

2) Determine the self-moment of the torso $[T_1]_A$, show that it is independent of point A;

3) Construct the torsor $[T]_A = a[T_1]_A + b[T_2]_A$ with a and b ∈ R;

4) What relation must a and b verify for the torsor $[T]_A$ to be a couple torsor;

5) Show that the couple torsor is independent of the point where it is measured;

6) Determine the simplest system of sliding vectors associated with the sum torsor:

$[T_1]_A + [T_2]_A$.

Corrected 03:

1) Centerline and Torso <u>Pitch</u> $[T_1]_A$

Centerline: It is defined by the set of P points such as: $\overrightarrow{OP} = \dfrac{\overrightarrow{R_1} \wedge \overrightarrow{M_{1A}}}{R_1^2} + \lambda \overrightarrow{R_1}$

$$\overrightarrow{OP} = \frac{1}{17}\begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix} \wedge \begin{pmatrix} 4 \\ -1 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix} = \frac{1}{17}\begin{pmatrix} -12 \\ -13 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{12}{17} - 3\lambda \\ -\frac{13}{17} + 2\lambda \\ -\frac{5}{17} + 2\lambda \end{pmatrix}$$

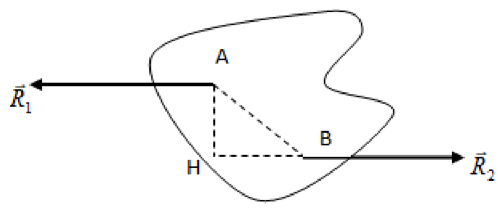
Torso pitch $[T_1]_A$: $\quad P_i = \dfrac{\overrightarrow{R_1} \cdot \overrightarrow{M_{1A}}}{R_1^2} = \dfrac{1}{17}(-3\vec{i} + 2\vec{j} + 2\vec{k}).(4\vec{i} - \vec{j} - 7\vec{k}) = -\dfrac{28}{17}$

2) Self Torso Moment $[T_1]_A$: $\vec{R_1}.\overrightarrow{M_{1A}} = (-3\vec{i} + 2\vec{j} + 2\vec{k}).(4\vec{i} - \vec{j} - 7\vec{k}) = -28$

The auto moment is independent of point A. Indeed, according to the transport formula we can write: $\overrightarrow{M_A} = \overrightarrow{M_B} + \overrightarrow{AB} \wedge \vec{R_1} \Rightarrow \vec{R_1}.\overrightarrow{M_A} = \vec{R_1}.\overrightarrow{M_B} + \vec{R_1}.(\overrightarrow{AB} \wedge \vec{R_1})$

$\vec{R_1}.\overrightarrow{M_A} = \vec{R_1}.\overrightarrow{M_B}$, it is clearly independent of point A

3) $[T]_A = a[T_1]_A + b[T_2]_A \Leftrightarrow [T]_A = \begin{cases} \vec{R} = a\vec{R_1} + b\vec{R_2} \\ \overrightarrow{M_A} = a\overrightarrow{M_{1A}} + b\overrightarrow{M_{2A}} \end{cases}$



$$[T]_A = \begin{cases} \vec{R} = -3(a-b)\vec{i} + 2(a-b)\vec{j} + 2(a-b)\vec{k} \\ \overrightarrow{M_{1A}} = 4(a+b)\vec{i} - (a-b)\vec{j} - 7(a-b)\vec{k} \end{cases}$$

4) Condition for $[T]_A$ to be a torque torsor: the result must be zero:

$\vec{R} = \vec{0} \Rightarrow$ a=b

The time in this case will be equal to: $\overrightarrow{M_{1A}} = 4(a+b)\vec{i} = 8a\vec{i}$

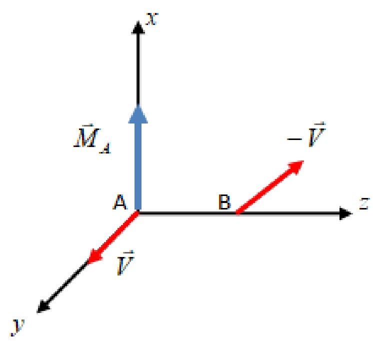
5) The moment of a couple torsor where the resultant $\vec{R_1}$ and $\vec{R_2}$ have the same module but opposite directions and applied to points A and B is written:

$$\overrightarrow{M_A} = \overrightarrow{OA} \wedge \vec{R_1} + \overrightarrow{OB} \wedge \vec{R_2} = \overrightarrow{OA} \wedge \vec{R_1} + \overrightarrow{OB} \wedge (-\vec{R})$$
$$\overrightarrow{M_A} = \overrightarrow{BA} \wedge \vec{R_1} = (\overrightarrow{BH} + \overrightarrow{HA}) \wedge \vec{R_1}$$
$$\overrightarrow{M_A} = \overrightarrow{HA} \wedge \vec{R_1} = -\overrightarrow{HA} \wedge \vec{R_1} = \overrightarrow{HA} \wedge \vec{R_2}$$

The moment of a couple is independent of the distance between points A and B, it depends only on the distance which separates the two support lines of the resultants. This distance is called the lever arm.

6) Simple system of sliding vectors associated with the torso sum: $[T_1]_A + [T_2]_A$.

The torso sum $[T]_A$ is given by: $[T]_A = \begin{cases} \vec{R} = \vec{0} \\ \overrightarrow{M_A} = 8\vec{i} \end{cases}$



The resultant can be decomposed into any two vectors of the same module and opposite direction, one of which is placed at point A, we then obtain:

$$\overrightarrow{M_A} = \overrightarrow{AA} \wedge \vec{V} + \overrightarrow{AB} \wedge (-\vec{V}) = \overrightarrow{AB} \wedge (-\vec{V}) = 5\vec{i}$$

System of two sliding vectors: (A, $\vec{V}$) and (B, -$\vec{V}$), such as $\vec{V}.\overrightarrow{M_A}$=0

Chapter II
Statics of Solids

Chapter II: Solid Statics

II.1 Introduction

Statics represents the field of rational mechanics that deals with the study of the equilibrium of mechanical systems considered at rest relative to the reference frame in which the observer is located. The mechanical system studied can represent any association of solid or fluid physical bodies, a point or a set of material points, a part, or the whole of a solid.

In this chapter, we address concepts related to material points, perfect solid bodies, force, the moment of a force, and external force torsors. We then provide the conditions for static equilibrium and the different types of connections and reactions. Finally, we explain some operations on forces concerning the reduction of a system of forces to a resultant and the decomposition of a force into several components.

We will see that static problems can be solved using graphical methods, analytical methods, or a combination of both methods.

II.2 Fundamental Concepts of Statics

II.2.1 Material Point

A material point is defined as a material particle that possesses mass and negligible dimensions under the conditions of the considered problem. The difference compared to the geometric point lies in the fact that the material point is assumed to contain a certain amount of concentrated matter.

II.2.2 Perfect Solid Body

A perfect solid body represents a theoretical model of the real solid, with natural and technological reality being more complex. A perfect solid body is made up of a set of material points that act on each other according to the principle of action and reaction equality and maintain the same distances between them under all circumstances, regardless of the applied external force systems. Therefore, a perfect solid body does not undergo any deformation.

II.2.3 Force

A force represents any interaction of one body on another. In mechanics, forces are used to model various mechanical actions (pressure, friction, contact actions, electrostatic force, electromagnetic force, etc.).

A force is represented by a force vector with the general properties of vectors: a point of application (A), a direction (or line of action) (Δ), a sense (from A to B), and a magnitude $\left\|\overrightarrow{AB}\right\|$.

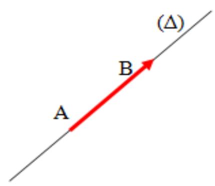


Figure II.1: Vector representation of a force

We can separate the action of a force on a body into two effects, external and internal:

- External forces can be either applied forces or reaction forces.
- Internal forces are resultants of stresses caused by external forces.

The unit of force is the Newton, which corresponds to the force that imparts an acceleration of 1 meter per second squared to a body with a mass of 1 kilogram.

II.2.4 Force Systems

A force system is defined as the set of forces $\vec{F_i}$ that act simultaneously on a material point or on a solid.

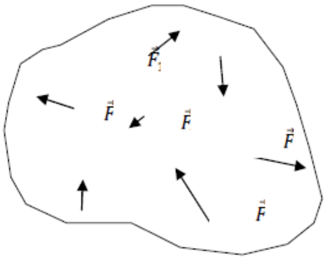


Figure II.2: Force System

Force systems are classified into three categories:

1. Reaction forces: If a solid body exerts a force on another body, the second body exerts an equal and opposite force on the first body.

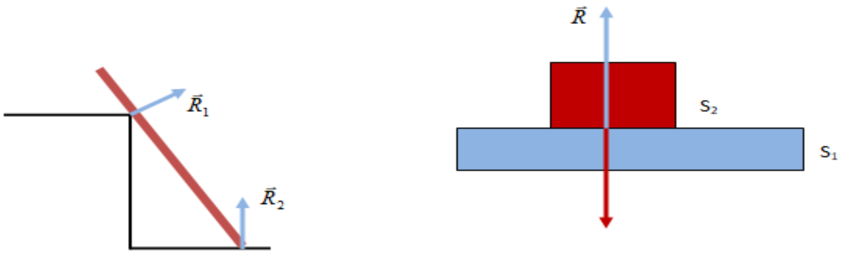


Figure II.3: Reaction Forces

2. Friction forces: Friction force exists when two real solids are in contact. Friction force always opposes the direction of movement.

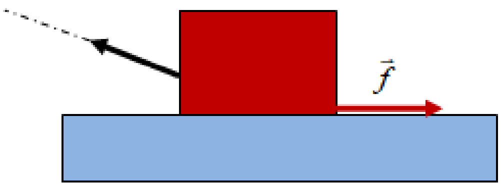


Figure II.4: Friction Forces

3. Tension forces: A force that pulls on an element of a body, such as the tension exerted by a string or spring.

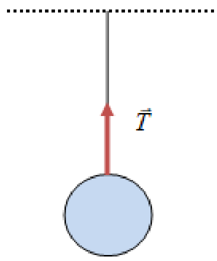


Figure II.5: Tension Forces

II.2.5 Operations on Force (Composition, Decomposition, Projection)

II.2.5.1 Geometric Decomposition of a Force

Consider a force $\vec{F}$ applied at the origin O of an orthonormal coordinate system. The components of this force are defined by:

$$\vec{F} = \vec{F_x} + \vec{F_y} + \vec{F_z} = \vec{F_x}\vec{i} + \vec{F_y}\vec{j} + \vec{F_z}\vec{k}$$

Such as $F_x = F.\cos\theta_x$, $F_y = F.\cos\theta_y$, $F_z = F.\cos\theta_z$

With: $F^2 = F^2_x + F^2_y + F^2_z$

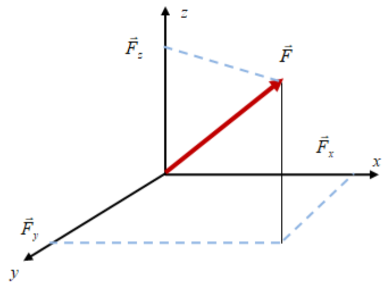


Figure II.6: Geometric decomposition of a force

The angles θ_x, θ_y, and θ_z are the three angles defined by the projection of the force on the three axes OX, OY, and OZ respectively.

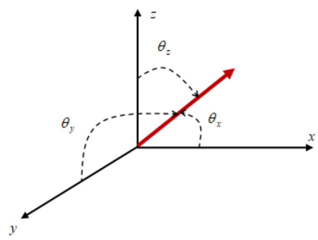


Figure II.7: Euler angles

The magnitude of the force can be expressed using cosines direction: $\frac{F}{\cos\theta_x} = \frac{F}{\cos\theta_y} = \frac{F}{\cos\theta_z}$

II.2.5.2 Resultant of Two Concurrent Forces

Given two forces $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$ applied at a point O of the solid, the resultant $\vec{R}$ can be determined from the parallelogram formed by these two forces (Figure).

The magnitude and direction of the resultant $\vec{R}$ are determined by the diagonal of the parallelogram constructed on these two forces.

$$\vec{R} = \overrightarrow{F_1} + \overrightarrow{F_2}$$

And its magnitude is: $R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\varphi}$

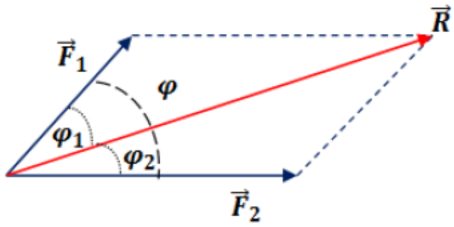


Figure II.8: Resulting from two forces

The direction is determined by:

$$\frac{F_1}{\sin\varphi_1} = \frac{F_2}{\sin\varphi_2} = \frac{F_3}{\sin\varphi_3} = \frac{R}{\sin\varphi}$$

II.3 Force Diagram

This is a graphical method used in the case of plane problems to determine the intensity of forces acting on a system in equilibrium. A body subjected to two forces is said to be in equilibrium if these forces are opposite in direction and have the same intensity and direction. If the body is subjected to three forces, for the body to be in equilibrium, the three forces must be concurrent. This relationship comes from the resultant equation derived from the fundamental principle of statics; the two sliders have the same central axis (the points of application of the forces are on a line collinear with the direction of the forces); this relationship comes from the moment equation derived from the same principle.

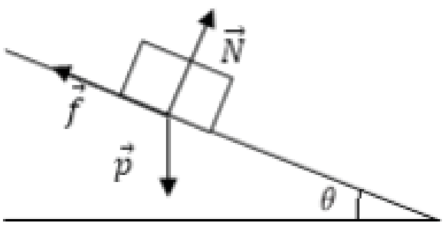


Figure II.9: Force diagram

II.4 Joints and Connections

II.4.1 Degrees of Freedom of a Free Solid

A solid is said to be free if it can move in any direction without restriction. Six independent directional movements are considered:

- Three degrees of translation
- Three degrees of rotation

The degrees of freedom are often presented in the form of a matrix where the columns give a type of movement (translation or rotation) and the rows the considered direction (x, y, or z).

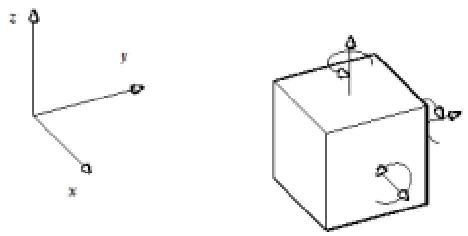


Figure II.10: Degrees of freedom

• Degrees of freedom refer to the number of independent parameters or values required to specify the state of an object.

• For a body to be in static equilibrium, all possible movements of the body need to be adequately restrained.

• Free body diagrams are used to identify the forces and moments that influence an object.

• Drawing a correct free-body diagram is the first and most important step in the process of solving an equilibrium problem.

II.4.2 Definition of a Connection

Connections are material bodies that oppose the movement of the solid. There is said to be a connection between two solids when one solid cannot move freely relative to another, reducing its degrees of freedom compared to a free body. The considered solids in mechanics can be free or connected depending on the case.

A solid is said to be free if it can move in any direction. For example, a stone thrown into space is a free solid. A solid is said to be connected if it can only move in determined directions or is constrained to remain immobile.

Material bodies that oppose the movement of the solid are called connections, and the forces they exert on the solid are called reaction forces.

II.4.3 Different Types of Connections and joints

Components in machinery, buildings etc., connect with each other and are supported in a number of different ways. In order to solve for the forces acting in such assemblies, one must be able to analyses the forces acting at such connections/supports.

II.4.3.1 Free Connection

This connection is essentially the absence of a connection; the solid is "left to itself" (e.g., a satellite in space or a projectile). There are six degrees of freedom and no transmitted contact force (no reaction).

II.4.3.2 Simple Support (Roller)

One of the most commonly occurring supports can be idealized as a roller support. Here, the contacting surfaces are smooth and the roller offers only a normal reaction force. This reaction force is labeled R_y , according to the x- y coordinate system shown. This is shown in the free-body diagram of–conventional the component.

The solid simply rests on a solid or a polished surface (horizontal, vertical, or inclined) (Figure II.11.a, b) or on a cylindrical roller (Figure II.11.c). The reaction of the surface is applied to the solid at the point of contact and directed along the normal to the support surface. It is called the normal reaction and is denoted by $\vec{R}$.

A simple support blocks movement in one direction and leaves two degrees of freedom.

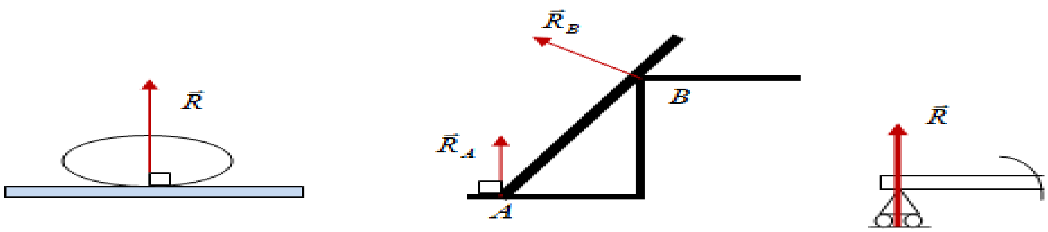


Figure II.11.a **Figure II.11.b** **Figure II.11.c**

Figure II.11: Point connection

II.4.3.3 Pin joint/hinged support (Double Supports)

Another commonly occurring connection is the pin joint. Here, the component is connected to a fixed hinge by a pin (going "into the page"). The component is thus constrained to move in one plane, and the joint does not provide resistance to this turning movement. The underlying support transmits a reaction force through the hinge pin to the component, which can have both normal (R_y) and tangential (R_x) components.

In practice, the solid body is sometimes articulated by:

- An articulated support (Figure II.12.a)
- A cylindrical articulation (sliding pivot connection, annular linear connection) (Figure II.8.b)
- Or a spherical articulation (ball-and-socket joint) (Figure II.8.c)

The solid is in contact with another solid with a cylindrical surface, blocking translations in two directions. The solid thus has a translation along the axis $\overrightarrow{Oz}$ and a rotation around the same axis. The reaction along the axis $\overrightarrow{Oz}$ of the articulation is zero.

$\overrightarrow{R_z} = \vec{0} \quad \vec{R} = \overrightarrow{R_x} + \overrightarrow{R_y}$

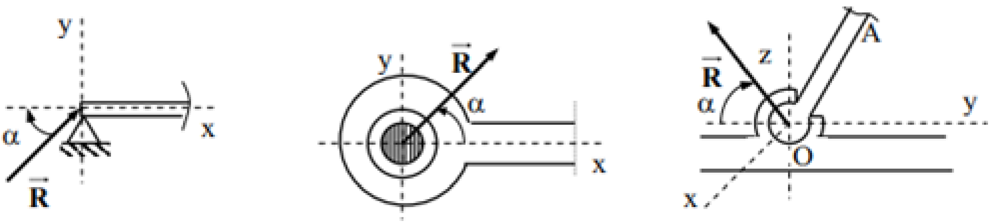


Figure II.12.a **Figure II.12.b** **Figure II.12.c**

Figure II.12: Articulated solids

II.4.3.4 Flexible Connection (String, Rope, Chain)

The reaction T is called tension. It is applied at the attachment point of the flexible link to the solid and directed along the flexible connection (the string, rope, chain, etc.) (Figure II.13).

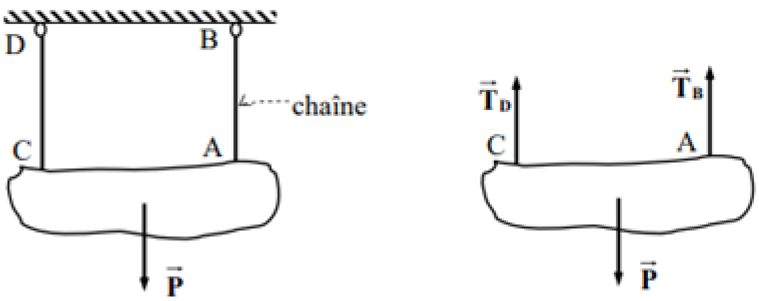


Figure II.13: Flexible connection

II.4.3.5 Fixed Connection

Finally, in Figure II.14 is shown a fixed (clamped) joint. Here the component is welded or glued and cannot move at the base. It is said to be cantilevered. The support in this case reacts with normal and tangential forces, but also with a couple of moment M, which resists any bending/turning at the base.

The fixed connection between two solids blocks their relative movements in all directions, preventing any movement (e.g., a cantilever beam). There are six reactions (three force components and three moment components). (Figure II.14)

The reactions are represented by components $\vec{R} = \overrightarrow{R_x} + \overrightarrow{R_y} + \overrightarrow{R_z}$ and a moment $\overrightarrow{M_A}$ that prevents the rotation of the solid.

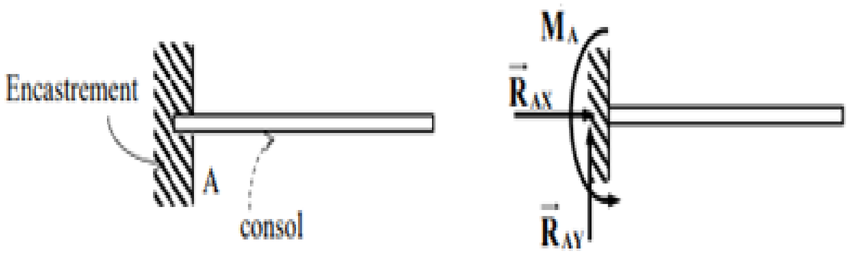


Figure II.14: Connection Embedding

II.5 Conditions for Equilibrium of a Rigid Body

II.5.1 Basic Conditions for Equilibrium of a Rigid Body

A solid body is in static equilibrium when several forces act simultaneously on it and these forces do not modify its state (state of rest or its state of movement).

For a solid body to be in static balance, the torso of external forces must be zero:

$$[T]_0 = \begin{pmatrix} \vec{R} \\ \vec{M}(\vec{F}_i) \end{pmatrix} = \begin{pmatrix} \vec{0} \\ \vec{0} \end{pmatrix}$$

For a rigid body subjected to external forces and in equilibrium, the following conditions must be met:

- The resultant force of all external forces acting on the body must be zero: $\vec{R} = \sum_{i=1}^{n} \vec{F}_i = \vec{0}$
- The resultant moment of all external forces about any point must be zero: $\sum_{i=1}^{n} \vec{M}(\vec{F}_i) = \vec{0}$

These two equilibrium conditions can be translated into six analytical equations by the projection of the elements of the force torso onto the axes of an orthonormal reference frame $R(O, \vec{i}, \vec{j}, \vec{k})$:

1. Three equations related to the resultant of external forces:

$$\vec{R} = \begin{cases} R_x = \sum_{n=1}^{n} \vec{F}_{xi} = \vec{0} \\ R_y = \sum_{n=1}^{n} \vec{F}_{yi} = \vec{0} \\ R_z = \sum_{n=1}^{n} \vec{F}_{zi} = \vec{0} \end{cases}$$

2. And three equations related to the moment of forces relative to point O:

$$\vec{M}_o(\vec{F}_i) = \begin{cases} \vec{M}_{ox}(\vec{F}_i) = \vec{0} \\ \vec{M}_{oy}(\vec{F}_i) = \vec{0} \\ \vec{M}_{oz}(\vec{F}_i) = \vec{0} \end{cases}$$

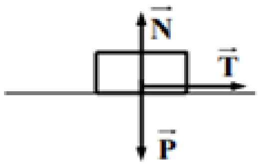
II.6 Equilibrium of Solids in the Presence of Friction

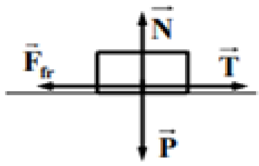
II.6.1 Sliding Friction

Sliding friction is the resistance that opposes the sliding of two rough surfaces in contact.

II.6.1.1 Experiment

Consider a solid of weight $\vec{P}$ resting on a horizontal surface. We apply a horizontal force $\vec{T}$ to this solid (Figure II.15.a).





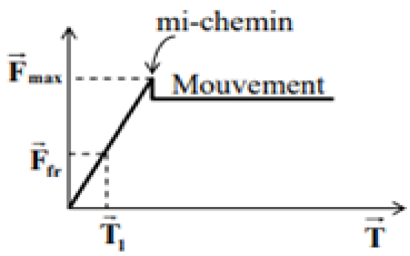


Figure II.15.a **Figure II.15.b** **Figure II.15.c**

1. **Polished contact surfaces:**

The weight force $\vec{P}$ is balanced by the reaction $\vec{N}$. In this case, no force opposes the driving force $\vec{T}$ (Figure II.15.a). The solid is in motion.

2. **Rough contact surfaces:**

The weight force $\vec{P}$ is balanced by the reaction $\vec{N}$. The solid can remain at rest; in this case, there is another force that opposes the movement of the solid in the same direction and opposite to $\vec{T}$ (Figure II.15.b). This force is called the sliding friction force $\overrightarrow{F_{fr}}$.

Increasing the force $\vec{T}$ gradually (Figure II.22.c). As long as the solid remains at rest, the force $\overrightarrow{F_{fr}}$ balances the driving force at each moment. The force $\overrightarrow{F_{fr}}$ increases with $\vec{T}$ up to a maximum value F_{max} ($\overrightarrow{F_{fr}} \leq F_{max}$) corresponding to the instant the solid begins to move. The maximum force corresponds to the limit case of the equilibrium of the solid, that is to say at the moment when it is halfway (in the transition zone) between rest and movement.

II.6.1.2 Static Friction Force

Sliding friction is a resisting force that acts in the tangent plane to the two contact surfaces, in the opposite direction to the driving force, and parallel to the contact surfaces. The friction force that acts when the body is stationary (at rest) is called the static friction force.

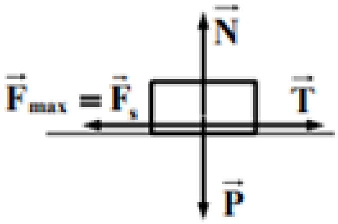


Figure II.16: Static Friction Force

According to Amontons-Coulomb's law, the maximum value of the static friction force $\overrightarrow{F_{max}}$ or $\overrightarrow{F_s}$ is proportional to the normal pressure $\vec{N}$ of the solid on the support surface: $\overrightarrow{F_{max}} = f_s . \vec{N}$

Where f_s coefficient of static friction that depends on the materials of the contact surfaces and their conditions. Some values of the sliding friction coefficient f_s for various materials are:

- Steel on ice: 0.027
- Steel on steel: 0.15
- Bronze on cast iron: 0.16
- Leather on cast iron: 0.28

II.6.1.3 Kinetic Friction Force

The friction force acting when a solid moves over another is the kinetic friction force $\overrightarrow{F_k}$. It is also proportional to the normal reaction $\vec{N}$: $\overrightarrow{F_k} = f_k . \vec{N}$

Where f_k is the kinetic friction coefficient. It depends on the speed of movement and is always less than the static friction coefficient ($f_k < f_s$).

II.6.2 Friction Angle

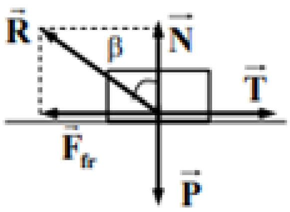


Figure II.17.a

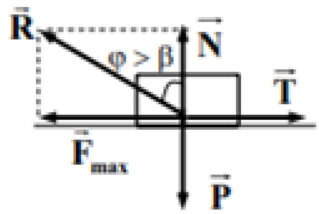


Figure II.17.b

When a solid body is at rest, the total reaction of a rough surface, considering the friction, is determined in magnitude and direction by the diagonal of the rectangle formed by the normal reaction $\vec{N}$ and the friction force $\overrightarrow{F_{fr}}$ (Figure II.24.a): $\vec{R} = \vec{N} + \overrightarrow{F_{fr}}$

The direction of $\vec{R}$ makes an angle β with $\vec{N}$ on the opposite side of $\vec{T}$. As $\vec{T}$ increases, the direction of $\vec{R}$ deviates more from the normal. The maximum deviation occurs when $F_{fr} = F_{max}$.

The maximum angle of deviation β is called the friction angle φ (Figure II.17.b) and is expressed as:

$$tg\varphi = \frac{F_{max}}{N} = \frac{f_s N}{N} = f_s \Rightarrow \varphi = arctg f_s$$

II.6.3 Rolling Friction

Rolling friction is the resistance that occurs when a solid rolls over another. Consider a cylindrical roller of weight $\vec{P}$ and radius $\vec{R}$ resting on a horizontal surface and acted upon by a driving force $\vec{T}$ at its center of gravity (Figure II.18.a).

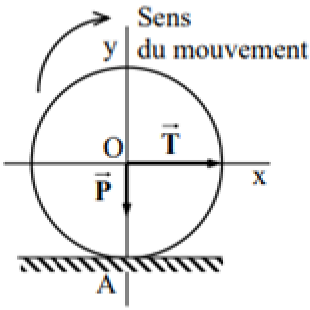


Figure II.18.a

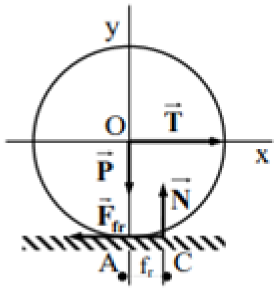


Figure II.18.b

The support surface deforms under the roller's weight, shifting the point of application of the reactions $\vec{N}$ and the friction force $\overrightarrow{F_{fr}}$ from point A to point C (Figure II.18.b). The equilibrium equations of the roller are:

$$\sum_{i=1}^{n} \overrightarrow{F_{ix}} = \vec{0} \Rightarrow T - F_{fr} = 0$$

$$\sum_{i=1}^{n} \overrightarrow{F_{iy}} = \vec{0} \Rightarrow N - P = 0$$

Where F_{fr} =T et N=P

The couple (F_{fr}, T) tends to put the roller in motion, while the torque (N, P) opposes the movement and tends to put the roller at rest. This last torque is called rolling resistance moment, m_r, it is equal to the moment of force N relative to point A.

$m_r = M_A(N)$

$\sum M_A(F) = M_A(N) - T\,R = 0$

Where $m_r = T\,R$

At the instant the solid starts to move, the resisting moment reaches its maximum value. Experiments show that this value is proportional to the normal reaction:

$(m_r)_{max} = f_r\ N$

The proportionality coefficient f_r is the rolling friction coefficient, measured in length units. At rest, we have: $mr \leq (m_r)_{max}$

$T\ R \leq fr.N$

Thus: $T \leq \dfrac{f_r}{R} N$

Generally, $\dfrac{f_r}{R}$ is much smaller than the sliding friction coefficient f_s, which is why when the rest is disturbed, the roller starts to roll over the support surface without sliding on it.

II.6.4 Friction of a Cable on a Pulley

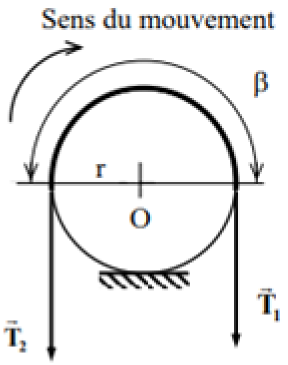
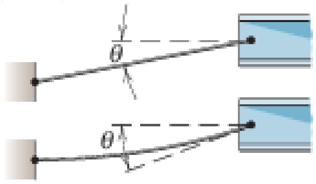
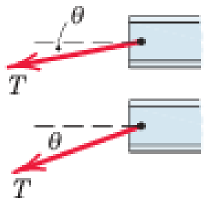
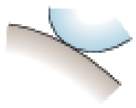
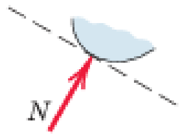
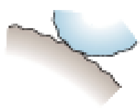
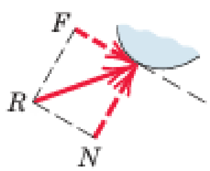
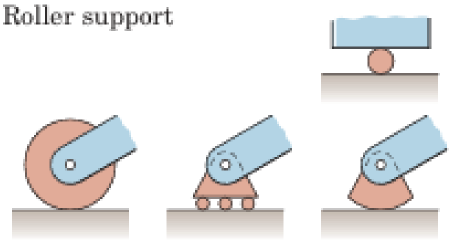
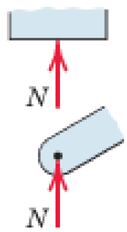
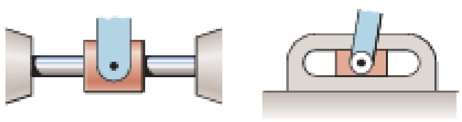
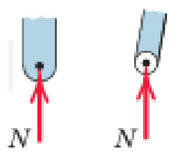
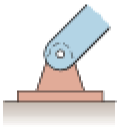
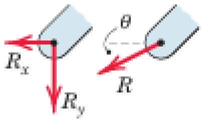
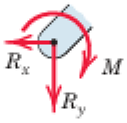
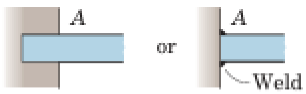
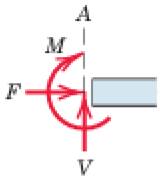
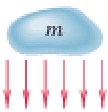
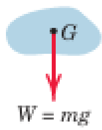
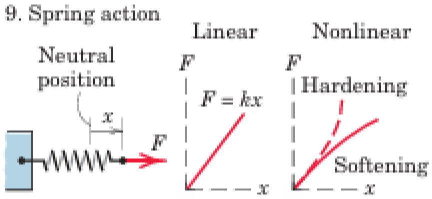
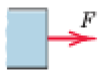


Figure II.19: Friction of a Cable on a Pulley

The relationship linking the two tensions T_1 and T_2 of a cable on a rough cylindrical surface (Figure II.26) is written as: $\dfrac{T_1}{T_2} = e^{f_s \beta}$

Where β is the angle of contact arc of the cable on the cylindrical surface, f_s is the static friction coefficient, and T_1 is always greater than T_2 ($T_1 > T_2$) depending on the direction of movement. The resultant friction force between the cable and the cylindrical surface is: $F = T_1 - T_2$

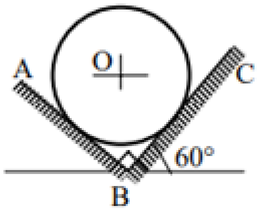
MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS	
Type of Contact and Force Origin	**Action on Body to Be Isolated**
1. Flexible cable, belt, chain, or rope Weight of cable negligible Weight of cable not negligible 	 Force exerted by a flexible cable is always a tension away from the body in the direction of the cable.
2. Smooth surfaces 	 Contact force is compressive and is normal to the surface.
3. Rough surfaces 	 Rough surfaces are capable of supporting a tangential component F (frictional force) as well as a normal component N of the resultant contact force R.
4. Roller support 	 Roller, rocker, or ball support transmits a compressive force normal to the supporting surface.
5. Freely sliding guide 	 Collar or slider free to move along smooth guides; can support force normal to guide only.

MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS (*cont.*)		
Type of Contact and Force Origin	**Action on Body to Be Isolated**	
6. Pin connection 	Pin free to turn  Pin not free to turn 	A freely hinged pin connection is capable of supporting a force in any direction in the plane normal to the pin axis. We may either show two components R_x and R_y or a magnitude R and direction θ. A pin not free to turn also supports a couple M.
7. Built-in or fixed support 		A built-in or fixed support is capable of supporting an axial force F, a transverse force V (shear force), and a couple M (bending moment) to prevent rotation.
8. Gravitational attraction 		The resultant of gravitational attraction on all elements of a body of mass m is the weight $W = mg$ and acts toward the center of the earth through the center mass G.
9. Spring action 		Spring force is tensile if spring is stretched and compressive if compressed. For a linearly elastic spring the stiffness k is the force required to deform the spring a unit distance.

Application Exercises

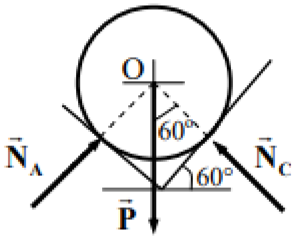
Exercise 01:

A homogeneous sphere O of weight 12 kN rests on two polished inclined planes AB and BC perpendicular to each other (Figure). Knowing that the plane BC makes an angle of 60° with the horizontal, determine the reactions of the two inclined planes on the sphere.



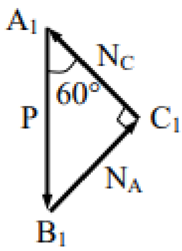
Solution 01: We remove the links of the sphere and replace them with the corresponding reactions (Figure 1). The sphere is in equilibrium under the action of three forces:

- The weight P acting vertically downwards.
- The reaction NA perpendicular to the plane AB towards the center O of the sphere.
- The reaction NC perpendicular to the plane BC towards the center O of the sphere.



The geometric equilibrium condition is based on the closed force polygon rule. We start by constructing the force polygon with the known force P. From an arbitrary point A_1, we draw the vector P (Figure 2). We place the origin of the next force, for example, N_A, at the end B_1 of the force vector $\vec{P}$. The magnitude of $\overrightarrow{N_A}$ is unknown.

Since the solid is in equilibrium, the force triangle P, N_A, N_C must be closed, so the end of the force vector $\overrightarrow{N_C}$ must coincide with the origin of vector $\vec{P}$, A_1.

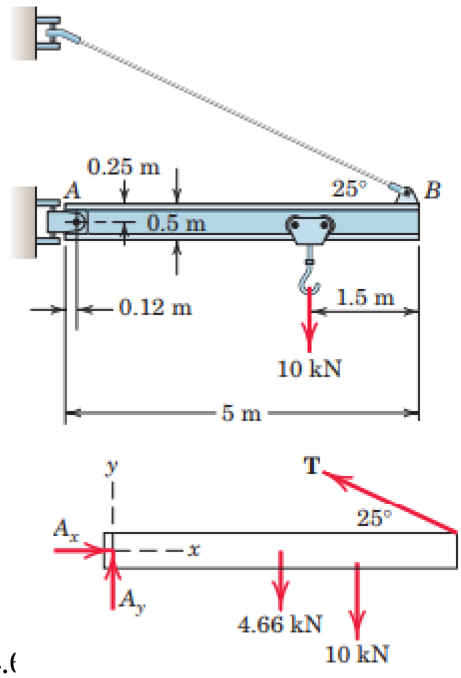


Applying the sine theorem on triangle $A_1B_1C_1$: $\dfrac{P}{\sin 90°} = \dfrac{N_A}{\sin 60°} = \dfrac{N_C}{\sin 30°}$

Thus: $N_A = \dfrac{\sin 60°}{\sin 90} P = 10.4KN$

$$N_C = \frac{\sin 30°}{\sin 90} P = 6KN$$

Exercise 02:



Free-body diagram

Determine the magnitude T of the tension in the supporting cable and the reaction (magnitude of the force) on the pin at A for the jib crane shown. The beam AB is a standard 0.5-m I-beam with a mass of 95 kg per meter of length.

Equating the sums of forces in the x and y directions to zero gives

$$\sum_{i=1}^{n} \vec{F}_{ix} = \vec{0} \Rightarrow A_x = 19.61\cos 25° = 0 \Rightarrow A_x = 17.77KN\,,$$

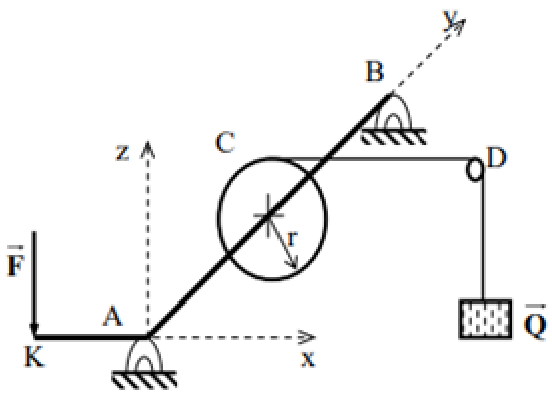
$$\sum_{i=1}^{n} \vec{F}_{iy} = \vec{0} \Rightarrow A_y + 19.61\sin 25° - 4.66 = 0 \Rightarrow A_y = 6.37KN$$

$$\sum_{i=1}^{n} \vec{M}_A(\vec{F}_i) = \vec{0} \Rightarrow (T\cos 25°)0.25 + (T\sin 25°)(5-0.12) - 10(5-1.5-0.12) - 4.6$$

From which $T = 19.61KN$

Exercise 03:

For the system shown in the Figure, determine the magnitude of the force $\vec{F}$ and the reactions at the cylindrical supports in A and B, knowing that friction at cylindrical surfaces C and D is negligible and we have: Q = 8 KN, r = 5 cm, AC = CB = 50 cm et AK = 40 cm.



Solution 02: We remove the links of the system shown in the Figure and replace them with the corresponding reactions. According to the linkage axiom, the system becomes free under the action of an arbitrary system of forces. Since friction in the pulley D is negligible, the tension in the cable CD remains constant: T=Q=8kN.

To determine the magnitude of the force $\vec{F}$ and the reactions at the cylindrical supports in A and B, we write the static equilibrium condition of the isolated solid body under an arbitrary system of forces. This condition is translated by the nullity of the external force tensor in A or B. The projection of the elements of this tensor on the axes is written as:

$$\sum_{i=1}^{n} \vec{F}_{ix} = \vec{0}\,,\ \sum_{i=1}^{n} \vec{F}_{iy} = \vec{0}\,,\ \sum_{i=1}^{n} \vec{F}_{iz} = \vec{0}$$

$$\sum_{i=1}^{n} \vec{M}_{Ax}(\vec{F}_i) = \vec{0}\,,\ \sum_{i=1}^{n} \vec{M}_{Ay}(\vec{F}_i) = \vec{0}\,,\ \sum_{i=1}^{n} \vec{M}_{Az}(\vec{F}_i) = \vec{0}$$

$$\sum_{i=1}^{n} \vec{M}_{Bx}(\vec{F}_i) = \vec{0}\,,\ \sum_{i=1}^{n} \vec{M}_{By}(\vec{F}_i) = \vec{0}\,,\ \sum_{i=1}^{n} \vec{M}_{Bz}(\vec{F}_i) = \vec{0}$$

$$\sum_{i=1}^{n} \vec{F}_{ix} = \vec{0} \Leftrightarrow -R_{Ax} + Q - R_{Bx} = 0 \qquad \textbf{(1)}$$

$$\sum_{i=1}^{n} \vec{F}_{iz} = \vec{0} \Leftrightarrow R_{Az} - F + R_{Bz} = 0 \qquad \textbf{(2)}$$

$$\sum_{i=1}^{n} \vec{M}_{Ax}(\vec{F}_i) = \vec{0} \Leftrightarrow R_{BZ} = 0 \qquad \textbf{(3)}$$

$$\sum_{i=1}^{n} \vec{M}_{Ay}(\vec{F}_i) = \vec{0} \Leftrightarrow F.KA - Q.r = 0 \quad \textbf{(4)}$$

$$\sum_{i=1}^{n} \vec{M}_{Az}(\vec{F}_i) = \vec{0} \Leftrightarrow R_{Bx.}.AB - Q.AC = 0 \quad \textbf{(5)}$$

Where

$$\sum_{i=1}^{n} \vec{M}_{Bx}(\vec{F}_i) = \vec{0} \Leftrightarrow -R_{Az}.AB + F.AB = 0 \; \textbf{(6)}$$

$$\sum_{i=1}^{n} \vec{M}_{Bz}(\vec{F}_i) = \vec{0} \Leftrightarrow R_{Bx}.AB - Q.BC = 0 \quad \textbf{(7)}$$

From the equation of equilibrium: F=1 KN

And from (4) it is determined: R_{Bz} = 0 KN

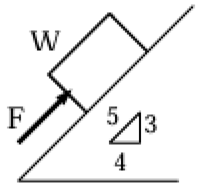
And from (5) it is determined: R_{Bx} = 4 KN

Thus, from (6), we determine: R_{AZ} = 1 KN

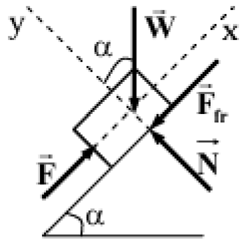
From (7), determine: R_{Ax} = 4 KN

The verification of equations (2) and (3), confirms the obtained results.

Exercise 04:



A force F=100 N is applied to a solid block with a weight W=300 N, placed on an inclined plane (Figure). The coefficient of static friction on the inclined plane at an angle α with respect to the horizontal is fs=0.25. Calculate the frictional force required to maintain equilibrium and check the equilibrium of the block if fs=0.4. What do you notice?



Let's start by calculating the magnitude of the frictional force capable of maintaining the block in equilibrium. By assuming that F_r is directed downwards and parallel to the inclined plane, we can draw the diagram of the isolated block (Figure 1.26b) and write the equilibrium equations:

$$\sum F_x = 0 \quad \Rightarrow \quad F - W\sin\alpha - F_r = 0 \quad (1)$$

$$\sum F_y = 0 \quad \Rightarrow \quad N - W\cos\alpha = 0 \quad (2)$$

Knowing that $\sin\alpha = 3/5$ and $\cos\alpha = 4/5$,

We replace F and W by their respective magnitudes, and we find the following after calculation:

$$F_r = -80\,\text{N} \quad \text{or} \quad F_r = 80\,\text{N directed upwards}$$

$$\text{and} \quad N = 240\,\text{N}$$

The force required to maintain equilibrium is an 80 N force directed upwards parallel to the inclined plane. Therefore, the block tends to move down the inclined plane.

The maximum frictional force:

The magnitude of the maximum frictional force is given by:

$$F_{\max} = f_s N$$

$$F_{\max} = 0.25 \times 240 = 60\,\text{N}$$

Since the value of the frictional force required to maintain equilibrium is $F_r = 80\,\text{N}$, which is greater than the maximum possible value $F_{\max} = 60\,\text{N}$, equilibrium cannot be maintained, and the block will slide down the inclined plane.

In the case where $f_s = 0.4$, the maximum frictional force is written as:

$$F_{\max} = 0.4 \times 240 = 96\,\mathrm{N}$$

In this case, $F_r = 80\,\mathrm{N} < F_{\max} = 96\,\mathrm{N}$, so the body can remain in equilibrium.

Chapter III

Kinematics of the Rigid Solid

Chapter III: Kinematics of the Rigid Body

III.1 Introduction

A rigid body is an idealization of a body that does not deform or change shape. Formally it is defined as a collection of particles with the property that the distance between particles remains unchanged during the course of motions of the body. Like the approximation of a rigid body as a particle, this is never strictly true. All bodies deform as they move. However, the approximation remains acceptable as long as the deformations are negligible relative to the overall motion of the body.

Kinematics of rigid bodies: relations between time and the positions, velocities, and accelerations of the particles forming a rigid body.

Classification of rigid body motions:
- Translation: rectilinear translation, curvilinear translation
- Rotation about a fixed axis;
- General plane motion;
- Motion about a fixed point;
- General motion;

III.2 Fundamental Assumptions

To study the motion of a material point P, or more generally a system of particles or solids, an observer must identify their position:

- In space;
- In time.

In classical kinematics, it is assumed that:

- space is Euclidean (three-dimensional);
- Time is absolute (independent of the observer).

III.3 Reference Frames

To fully study kinematic motion, the observer must define:

- a spatial reference frame linked to the observer with an origin O and an orthonormal basis $(\vec{i}, \vec{j}, \vec{k})$ forming the trihedron $(O, \vec{i}, \vec{j}, \vec{k})$, which fully defines the spatial reference frame;
- a time reference (time scale) with an origin and a unit of measurement. In the MKSA system, the second is the unit of time.

The spatial reference frame and the time reference together define the <space-time> reference frame noted as (R). In this frame, at a given moment by the clock, the position of a point r (t) is defined by its coordinates x (t), y (t), z (t) such that:

$$\overrightarrow{Or} = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

The position of point P is known instantaneously in both space and time.

III.4 Motion Relative to Translating Axes

III.4.1 Trajectory

Let point M be identified in a fixed reference frame R (O, $\vec{i}$, $\vec{j}$, $\vec{k}$). Its position is given at each instant t by the vector (Figure III.1):

$$\overrightarrow{r(t)} = \overrightarrow{OM} = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k},$$

The vector $\overrightarrow{r(t)}$ has components in the fixed reference frame at instant t. $\overrightarrow{r(t)} = \begin{cases} x(t) \\ y(t) \\ z(t) \end{cases}$

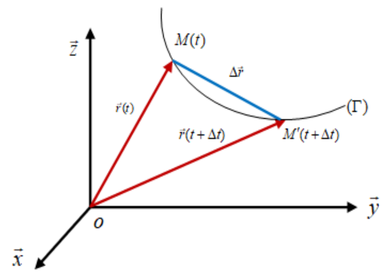


Figure III.1: Trajectory of a point

The displacement of point M in space is given by the parametric equations of coordinates (x, y, z) as functions of time. By eliminating the time parameter, we obtain the trajectory described by this point in space.

$\overrightarrow{r(t)}$ = M(t): position of point M in R (O, $\vec{i}$, $\vec{j}$, $\vec{k}$) at instant t.

$\overrightarrow{r(t+\Delta t)}$ =M(t+Δt): position of point M in R (O, $\vec{i}$, $\vec{j}$, $\vec{k}$) at instant t+Δt.

The displacement vector from $\overrightarrow{r(t)}$ to $\overrightarrow{r(t+\Delta t)}$ is given by $\Delta\overrightarrow{r(t)} = \overrightarrow{r(t+\Delta t)} - \overrightarrow{r(t)}$.

The positions occupied by point M in space describe a trajectory (Γ) with respect to the chosen reference frame R (O, $\vec{i}$, $\vec{j}$, $\vec{k}$).

III.4.2 Velocity Vector

The material point moves from position M(t) to position M(t+Δt) during the time interval Δt at an average speed:

$$\overrightarrow{V_m} = \frac{\overrightarrow{MM'}}{\Delta t} = \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \frac{\overrightarrow{\Delta r(t)}}{\Delta t}$$

The instantaneous velocity vector is obtained when: Δt →0, defined as:

$$\vec{V} = \lim_{\Delta t \to 0} \overrightarrow{V_m} = \lim_{\Delta t \to 0} \frac{\overrightarrow{\Delta r(t)}}{\Delta t} = \frac{d\vec{r}(t)}{\Delta t}$$

This vector is always tangent to the trajectory and directed in the direction of motion.

III.4.3 Acceleration Vector

The derivative of the velocity vector in the same reference frame R (O, $\vec{i}$, $\vec{j}$, $\vec{k}$) gives the instantaneous acceleration of point M:

$$\overrightarrow{\gamma_m} = \frac{\vec{V}(t+\Delta t) - \vec{V}(t)}{\Delta t} = \frac{\Delta\vec{V}(t)}{\Delta t}$$

The instantaneous acceleration is:

$$\vec{\gamma} = \lim_{\Delta t \to 0} \overrightarrow{\gamma_m} = \lim_{\Delta t \to 0} \frac{\overrightarrow{\Delta V(t)}}{\Delta t} = \frac{d\vec{V}(t)}{\Delta t} = \frac{d^2\vec{r}(t)}{dt^2}$$

The two kinematic vectors help to understand the nature of the motion and to predict the different phases, depending on whether the velocity vector is in the same or opposite direction to the acceleration vector.

III.5 Coordinate Systems

The material point M can be identified in space within a fixed reference frame (R) centered at O by three different but related types of coordinates:

- Cartesian: (x, y, z) with unit vectors of the reference frame ($\vec{i}$, $\vec{j}$, $\vec{k}$);
- Cylindrical: (r, θ, z) with unit vectors of the reference frame ($\overrightarrow{u_r}$, $\overrightarrow{u_\theta}$, $\vec{k}$);
- Spherical: (r, θ, φ) with unit vectors of the reference frame ($\overrightarrow{e_r}$, $\overrightarrow{e_\theta}$, $\overrightarrow{e_\varphi}$).

These three types of coordinates allow the description of all types of motions of point M in space.

III.5.1 Cartesian Coordinates

Also called rectangular coordinates. If point M is identified in R (O, $\vec{i}$, $\vec{j}$, $\vec{k}$) by the Cartesian coordinates (x, y, z), which depend on time, the position vector $\overrightarrow{OM}$ would be written as: $\overrightarrow{OM} = x\vec{i} + y\vec{j} + z\vec{k}$;

$$\overrightarrow{OM} = x\vec{i} + y\vec{j} + z\vec{k} \ ; \ \overrightarrow{OM} = \begin{cases} x \\ y \\ z \end{cases} ; \ OM = \sqrt{x^2 + y^2 + z^2}$$

The velocity and acceleration vectors are deduced by the first and second derivatives:

$$\vec{V}(t) = \frac{d\overrightarrow{OM}(t)}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} \text{ ; written as: } \vec{V}(t) = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}$$

With: $\left|\vec{V}(t)\right| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$

$$\vec{\gamma}(t) = \frac{d\vec{V}(t)}{dt} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{d^2t}\vec{k}$$; written as: $\vec{\gamma}(t) = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k}$

With: $\left|\vec{\gamma}(t)\right| = \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}$

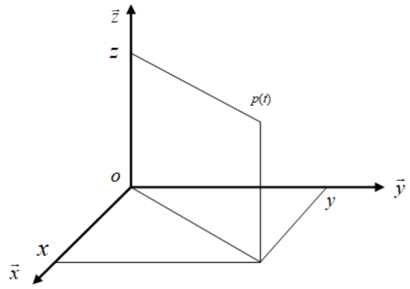


Figure III.2: Cartesian coordinates

III.5.2 Cylindrical Coordinates

If point M is identified by the cylindrical coordinates (r, θ, z), which depend on time, in a reference frame R (O, $\overrightarrow{u_r}, \overrightarrow{u_\theta}, \vec{k}$), the position vector would be written as: $\overrightarrow{OM} = r\overrightarrow{u_r} + z\vec{k}$

In the reference frame R (O, $\overrightarrow{u_r}, \overrightarrow{u_\theta}, \vec{k}$), the vector $\overrightarrow{OM}$ is written as: $\overrightarrow{OM} = \begin{cases} r\cos\theta \\ r\sin\theta \\ z \end{cases}$

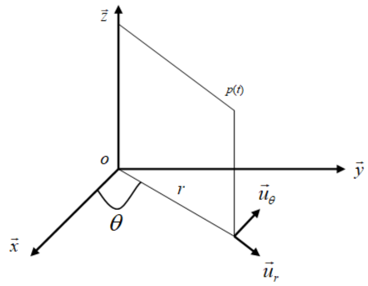


Figure III.3: Cylindrical coordinates

$$\vec{V} = \frac{d\overrightarrow{OM}}{dt} = \dot{r}\overrightarrow{u_r} + r\frac{d\overrightarrow{u_r}}{dt} + \dot{z}\vec{k}$$

With: $\frac{d\overrightarrow{u_r}}{dt} = \frac{d\overrightarrow{u_r}}{d\theta}.\frac{d\theta}{dt} = \dot{\theta}\overrightarrow{u_\theta}$, we obtain: $\vec{V} = \dot{r}\overrightarrow{u_r} + r\dot{\theta}\overrightarrow{u_\theta} + \dot{z}\vec{k}$

$V_r = \dot{r}$, $V_\theta = r\dot{\theta}$, $V_z = \dot{z}$

The acceleration is determined by:

$$\vec{\gamma} = \frac{d^2\overrightarrow{OM}}{dt^2} = \frac{d\vec{V}}{dt} = \frac{d(\dot{r}\overrightarrow{u_r})}{dt} + \frac{d(r\dot{\theta}\overrightarrow{u_\theta})}{dt} + \ddot{z}\vec{k}$$

$$\vec{\gamma} = \ddot{r}\overrightarrow{u_r} + \dot{r}\frac{d\overrightarrow{u_r}}{dt} + \dot{r}\dot{\theta}\overrightarrow{u_\theta} + r\ddot{\theta}\overrightarrow{u_\theta} + r\dot{\theta}\frac{d\overrightarrow{u_\theta}}{dt} + \ddot{z}\vec{k}$$

We have: $\frac{d\overrightarrow{u_r}}{dt} = \frac{d\overrightarrow{u_r}}{d\theta}.\frac{d\theta}{dt} = \dot{\theta}\vec{u}_\theta$; $\frac{d\overrightarrow{u_\theta}}{dt} = \frac{d\overrightarrow{u_\theta}}{d\theta}.\frac{d\theta}{dt} = -\dot{\theta}\vec{u}_r$;

The expression for acceleration becomes:

$\vec{\gamma} = (\ddot{r} - r\dot{\theta}^2)\vec{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{u}_\theta + \ddot{z}\vec{k}$; Where $\gamma = \sqrt{(\ddot{r} - r\dot{\theta}^2)^2 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})^2 + \ddot{z}^2}$;

$\gamma_r = (\ddot{r} - r\dot{\theta}^2)$; $\gamma_\theta = (r\ddot{\theta} + 2\dot{r}\dot{\theta})$; $\gamma_z = \ddot{z}$

III.5.3 Spherical Coordinates

In the reference frame R (O,$\vec{i}$, $\vec{j}$, $\vec{k}$), the vector $\overrightarrow{OM}$ has components: $\overrightarrow{OM} = \begin{cases} r\cos\varphi\cos\theta \\ r\cos\varphi\sin\theta \\ r\sin\varphi \end{cases}$

In spherical coordinates, it is written as: $\overrightarrow{OM} = OM\vec{e}_r = r\vec{e}_r$

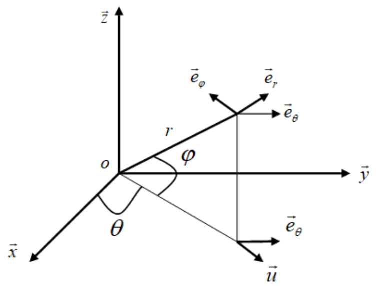


Figure III.4: Spherical coordinates

With:

$\vec{e}_r = \cos\varphi\vec{u} + \sin\varphi\vec{k}$; $\vec{e}_\varphi = -\sin\varphi\vec{u} + \cos\varphi\vec{k}$

$\vec{u} = \cos\varphi\vec{e}_r - \sin\varphi\vec{e}_\varphi$; $\frac{d\vec{u}}{d\theta} = \vec{e}_\theta$; $\frac{d\vec{e}_\theta}{d\theta} = -\vec{u}$; $\frac{d\vec{e}_r}{d\varphi} = -\vec{e}_\varphi$; $\frac{d\vec{e}_\varphi}{d\varphi} = -\vec{e}_r$

So: $\frac{d\vec{e}_r}{dt} = -\dot{\varphi}\sin\varphi\vec{u} + \cos\varphi\frac{d\vec{u}}{dt} + \dot{\varphi}\cos\varphi\vec{k} = \dot{\theta}\cos\varphi\vec{e}_\theta - \dot{\varphi}\sin\varphi\vec{u} + \dot{\varphi}\cos\varphi\vec{k}$

$\frac{d\vec{e}_r}{dt} = \dot{\theta}\cos\varphi\vec{e}_\theta + \dot{\varphi}(-\sin\varphi\vec{u} + \cos\varphi\vec{k}) = \dot{\theta}\cos\varphi\vec{e}_\theta + \dot{\varphi}\vec{e}_\varphi$

The velocity of point M is deduced by: $\vec{V} = \frac{d\overrightarrow{OM}}{dt} = \frac{d(r\overrightarrow{e_r})}{dt} = \dot{r}\vec{e}_r + r\frac{d\vec{e}_r}{dt} = \dot{r}\vec{e}_r + r\dot{\theta}\cos\varphi\vec{e}_\theta + r\dot{\varphi}\vec{e}_\varphi$

$$\vec{V} = \begin{cases} V_r = \dot{r} \\ V_\theta = r\dot{\theta}\cos\varphi \\ V_\varphi = r\dot{\varphi} \end{cases}$$

The acceleration is easily deduced by differentiating the velocity expression with respect to time:

$$\vec{\gamma} = \frac{d\vec{V}}{dt} = \frac{d(\dot{r}\vec{e_r})}{dt} + \frac{d(r\dot{\theta}\cos\varphi\vec{e}_\theta)}{dt} + \frac{d(r\dot{\varphi}\vec{e}_\varphi)}{dt}$$

$$(1): \frac{d(\dot{r}\vec{e_r})}{dt} = \ddot{r}\vec{e}_r + \dot{r}(\dot{\theta}\cos\varphi\vec{e}_\theta + \dot{\varphi}\vec{e}_\varphi) = \ddot{r}\vec{e}_r + \dot{r}\dot{\theta}\cos\varphi\vec{e}_\theta + \dot{r}\dot{\varphi}\vec{e}_\varphi$$

$$(2): \frac{d(r\dot{\theta}\cos\varphi\vec{e}_\theta)}{dt} = \dot{r}\dot{\theta}\cos\varphi\vec{e}_\theta + r\ddot{\theta}\cos\varphi\vec{e}_\theta - r\dot{\theta}\dot{\varphi}\sin\varphi\vec{e}_\theta + r\dot{\theta}\cos\varphi\frac{d\vec{e}_\theta}{dt}$$

$$\frac{d\vec{e_\theta}}{dt} = \frac{d\vec{e}_\theta}{d\theta}.\frac{d\theta}{dt} = -\dot{\theta}\vec{u} = -\dot{\theta}(\cos\varphi\vec{e}_r - \sin\varphi\vec{e}_\varphi)$$

$$\frac{d(r\dot{\theta}\cos\varphi\vec{e}_\theta)}{dt} = \dot{r}\dot{\theta}\cos\varphi\vec{e}_\theta + r\ddot{\theta}\cos\varphi\vec{e}_\theta - r\dot{\theta}\dot{\varphi}\sin\varphi\vec{e}_\theta - r\dot{\theta}^2\cos\varphi(\cos\varphi\vec{e}_r - \sin\varphi\vec{e}_\varphi)$$

$$\frac{d(r\dot{\theta}\cos\varphi\vec{e}_\theta)}{dt} = -r\dot{\theta}^2\cos^2\varphi\vec{e}_r + (\dot{r}\dot{\theta}\cos\varphi - r\ddot{\theta}\cos\varphi - r\dot{\theta}\dot{\varphi}\sin\varphi)\vec{e}_\theta + r\dot{\theta}^2\cos\varphi\sin\varphi\vec{e}_\varphi$$

$$(3): \frac{d(r\dot{\varphi}\vec{e}_\varphi)}{dt} = \dot{r}\dot{\varphi}\vec{e}_\varphi + r\ddot{\varphi}\vec{e}_\varphi + r\dot{\varphi}\frac{d\vec{e}_\varphi}{dt} = \dot{r}\dot{\varphi}\vec{e}_\varphi + r\ddot{\varphi}\vec{e}_\varphi + r\dot{\varphi}\frac{d\vec{e}_\varphi}{d\varphi}.\frac{d\varphi}{dt}$$

As then: $\frac{d\vec{e}_\varphi}{dt} = -\dot{\varphi}\vec{e}_r$ then $\frac{d(r\dot{\varphi}\vec{e}_\varphi)}{dt} = \dot{r}\dot{\varphi}\vec{e}_\varphi + r\ddot{\varphi}\vec{e}_\varphi - r\dot{\varphi}^2\vec{e}_r$

Summing the three terms, we get:

$$\gamma_r = \ddot{r} - r\dot{\varphi}^2 - r\dot{\theta}^2\cos^2\varphi$$

$$\gamma_\theta = -\dot{r}\dot{\theta}\cos\varphi + \dot{r}\dot{\theta}\cos\varphi + r\ddot{\theta}\cos\varphi - r\dot{\theta}\dot{\varphi}\sin\varphi = \frac{\cos\varphi}{r}.\frac{d}{dt}(r^2\dot{\theta}) - r\dot{\theta}\dot{\varphi}\sin\varphi$$

$$\gamma_\varphi = \dot{r}\dot{\varphi} + r\dot{\theta}^2\cos\varphi\sin\varphi + \dot{r}\dot{\varphi} + r\ddot{\varphi} = \frac{1}{r}.\frac{d}{dt}(r^2\dot{\varphi}) + r\dot{\theta}^2\sin\varphi\cos\varphi$$

III.6 Special motions

III.6.1 Circular Trajectory motion

A particle M is in circular motion if at any instant t, it is located at a point P on a circle (c) of radius "a" and center O. Choose an orthonormal reference frame with origin O and unit vectors $\vec{i}$ and $\vec{j}$, located in the plane of the circular trajectory.

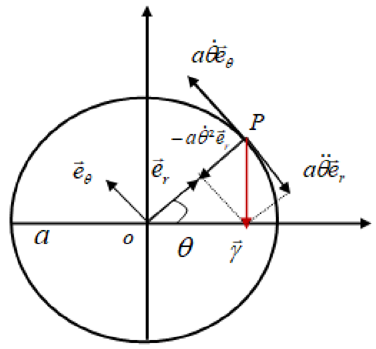


Figure III.5: Circular motion

Choose a circle in the (Oxy) plane so that its center coincides with that of the reference frame. Point P on the circle is identified by two coordinates:

- The radius a of the circle and the angle $\theta = (\overrightarrow{Ox}, \overrightarrow{OP})$ that the vectors $\overrightarrow{OP}$ make with the axis $\overrightarrow{Ox}$.

Let $\vec{e}_r$ the vector be defined by: $\vec{e}_r = \dfrac{\overrightarrow{OP}}{OP}$, then we have: $\overrightarrow{OP} = OP.\vec{e}_r$

The unit vector $\vec{e}_r$ changes direction with the angle θ: hence $\dfrac{d\vec{e}_r}{d\theta} = \vec{e}_\theta$ and $\dfrac{d\vec{e}_\theta}{d\theta} = -\vec{e}_r$

The radius of curvature is constant here; the velocity of point P is given by the derivative of the position vector:

$$\vec{V}(P) = \frac{d\overrightarrow{OP}}{dt} = a\frac{d\vec{e}_r}{dt} = a\frac{d\vec{e}_r}{d\theta}.\frac{d\theta}{dt} = a\dot{\theta}\vec{e}_\theta$$

The acceleration of point P is deduced by:

$$\vec{\gamma}(P) = \frac{d\overrightarrow{V(P)}}{dt} = -a\dot{\theta}^2\vec{e}_r + a\ddot{\theta}\vec{e}_\theta$$

$\dot{\theta} = \omega$: angular velocity of point P;

$\ddot{\theta} = \dot{\omega}$: angular acceleration of point P.

The velocity of point P is tangent to the circle with an algebraic value: $\vec{V}(P) = a\dot{\theta}\vec{e}_\theta$

The acceleration of point P has two components: one tangential: $\gamma_t = a\ddot{\theta} = a\dot{\omega}$, the other normal: $\gamma_n = -a\dot{\theta}^2 = -a\omega^2$.

Note that the normal acceleration vector γ_n is always opposite to the position vector $\overrightarrow{OP}$: $\vec{\gamma}_n = -a\dot{\theta}^2\vec{e}_r = -\omega^2\overrightarrow{OP}$

Knowing the angular velocity and acceleration, we can determine the nature of the motion:

- If $\dot{\theta}\,\ddot{\theta} > 0$, the motion is accelerated;
- If $\dot{\theta}\,\ddot{\theta} < 0$, the motion is decelerated;
- If $\ddot{\theta} = 0 \Rightarrow \dot{\theta}$=Cte, the motion is uniform; the tangential acceleration is zero, but the normal acceleration is not.

III.6.2 Helical Trajectory motion

A point P moves on a helical trajectory in a reference frame R $(O, \vec{i}, \vec{j}, \vec{k})$ if it describes a right helix drawn on a cylinder of radius a. The Cartesian coordinates of point P in this reference frame are given by the parametric equations as functions of time t in the following form:

$$\overrightarrow{OP} = \begin{cases} x(\theta) = a\cos\theta(t) \\ y(\theta) = a\sin\theta(t) \\ z(\theta) = b\theta(t) \end{cases} \qquad \text{a: radius of the helix}$$

The angle θ plays the same role as in cylindrical or polar coordinates. The parameter b = Cte is called the pitch of the helix. Note that when the angle θ increases by 2π, the positions x and y do not change, but along the vertical z-axis, there is a displacement of: 2π b ;

$$x(\theta + 2\pi) = x(\theta)\ ;\ y(\theta + 2\pi) = y(\theta)$$

$$z(\theta + 2\pi) = b(\theta + 2\pi) = b\theta + 2\theta b = z(\theta) + 2\pi b$$

The position vector of point P in the reference frame R $(O, \vec{i}, \vec{j}, \vec{k})$ is given by: $\overrightarrow{OP} = a\vec{e}_r = z\vec{k} = a\vec{e}_r + b\theta\vec{k}$

The velocity and acceleration vectors are written as:

$$\vec{V}(P) = a\dot{\theta}\vec{e}_\theta + b\dot{\theta}\vec{k} = V_\theta\vec{e}_\theta + V_z\vec{k}$$

$$\vec{\gamma}(P) = -a\dot{\theta}^2\vec{e}_r + a\ddot{\theta}\vec{e}_\theta + b\ddot{\theta}\vec{k}$$

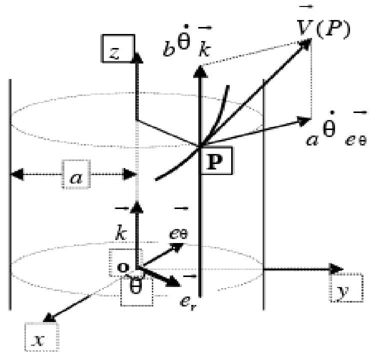


Figure III.6: Helical trajectory motion

Note that the ratio between the components of the velocity along the unit vectors $\vec{e}_\theta$ and $\vec{k}$ is independent of the angle θ.

$$\frac{V_z}{V_\theta} = \frac{b\dot{\theta}}{a\dot{\theta}} = \frac{b}{a}$$

This expression indicates that any tangent at a point P on the helix makes a constant angle with the vertical passing through point P and parallel to the vector. The helical motion is uniform if the angular velocity of rotation is constant, hence independent of the time parameter ($\dot{\theta}$= ω=cte). In this case, the velocity and acceleration expressions are: $\vec{V}(P) = a\omega\vec{e}_\theta + b\omega\vec{k}$

With $V(P) = \omega\sqrt{(a^2 + b^2)}$

$\vec{\gamma}(P) = -a\omega^2\vec{e}_r$ the acceleration is directed inward of the curvature. Previously, in curvilinear motions, it was shown that the acceleration of point P is written as $\vec{\gamma}(P) = \frac{dV}{dt}\vec{\tau} + \frac{V^2}{\rho}\vec{n}$, where the unit vectors $\vec{\tau}$ and $\vec{n}$ are the tangential and normal vectors at point P on the curve.

Applying this relation in the case of uniform helical motion where $\vec{\tau} = \vec{e}_\theta$ and $\vec{n} = -\vec{e}_r$ are the tangential and normal vectors at point P on the curve, we get:

$\vec{\gamma}(P) = -a\omega^2\vec{e}_r = \frac{V^2}{\rho}\vec{n} \Rightarrow -a\omega^2\vec{e}_r = -\frac{V^2}{\rho}\vec{e}_r \Leftrightarrow a\omega^2 = \frac{V^2}{\rho}$ replacing the velocity by its expression, we obtain: $a\omega^2 = \frac{\omega^2(a^2 + b^2)}{\rho} \Rightarrow \rho = \frac{(a^2 + b^2)}{a} = a + \frac{b^2}{a}$

Since the normal at P is always directed inward of the curvature, the center of curvature C can be easily determined by writing the following relation: $\overrightarrow{PC} = -\rho\vec{e}_r$.

III.7 Kinematics of the Rigid Body

A perfect rigid body (S) is a set of material elements whose mutual distances do not vary over time. Consequently, the velocities between these points are not independent. Hence, the kinematics of the rigid body deals with the distribution of velocities of points within a body independently of the causes that generated the motion of the solid.

The mechanics of solids allow us to study the behavior of solids and determine all the kinematic parameters of all its points regardless of the nature of the motion. The transport formula allows, by knowing the speed of a single point of the solid, to easily deduce the speed of all points of the solid. The objective of the kinematics of the solid is to know the position, speed, and acceleration of all points of the solid relative to a determined frame of reference.

III.7.1 Concept of Frames and Reference Systems

To study the motion of a solid or a system composed of several solids, it is essential to locate the position of each point as well as the kinematic vectors in space and time. In classical kinematics, we consider that space is three-dimensional Euclidean and time is absolute and independent of the observer. To locate the solid, the observer defines:

• A spatial frame defined by an origin O and an orthonormal basis $(\vec{x}_0, \vec{y}_0, \vec{z}_0)$. The trihedron $(O, \vec{x}_0, \vec{y}_0, \vec{z}_0)$ completely defines the spatial frame in which the coordinates of all points of the solid can be expressed.
• A time frame (also called a time scale) with an origin and a time unit.

In the MKSA system, the unit of time is the second.

These two frames define a space-time frame called a reference frame or simply a frame in classical kinematics. We then choose an arbitrary point O_s on the solid. The position of this point is given at each instant by the position vector $\overrightarrow{OO_s}$ expressed in the frame $R(O, \vec{x}_0, \vec{y}_0, \vec{z}_0)$. The coordinates of the point O_s depend on time and allow us to know at any moment the position of the frame $R(O, \vec{x}_s, \vec{y}_s, \vec{z}_s)$ linked to the solid. The transition from the frame $R(O, \vec{x}_0, \vec{y}_0, \vec{z}_0)$ to the frame $R(O, \vec{x}_s, \vec{y}_s, \vec{z}_s)$ linked to the solid is determined by the transition matrix, which expresses the unit vectors $(O, \vec{x}_0, \vec{y}_0, \vec{z}_0)$ in terms of the unit vectors $(O, \vec{x}_s, \vec{y}_s, \vec{z}_s)$. This transition matrix is expressed in terms of Euler angles. The orientation of the frame linked to the solid is independent of the choice of the point O_s.

The set of translation and rotation parameters constitute the situation parameters or degrees of freedom of the solid in space relative to the frame $R(O, \vec{x}_0, \vec{y}_0, \vec{z}_0)$. If the number of parameters is equal to 6 (3 rotations and 3 translations), the solid is said to be completely free in $R(O, \vec{x}_0, \vec{y}_0, \vec{z}_0)$. If the number of parameters is less than 6, the solid is said to be constrained or subjected to constraints where certain parameters do not vary over time.

III.7.2 Notation Systems

In the study of kinematics, we adopt the following notation:

Let $R_i(O, \vec{x}_i, \vec{y}_i, \vec{z}_i)$ be a frame linked to the observer and P a point of the solid:

- $\overrightarrow{O_iP}$: Position vector of point P relative to frame R_i;
- $\vec{V}^i(P) = \dfrac{d^i \overrightarrow{O_iP}}{dt}$: Speed of point P relative to frame R_i;
- $\vec{\gamma}^i(P) = \dfrac{d^i \vec{V}^i(P)}{dt}$: Acceleration of point P relative to frame R_i.

The kinematic parameters are always linked to the frame. The kinematic parameters (velocity and acceleration vectors) of the points of the solid are studied in a frame $R_i(O, \vec{x}_i, \vec{y}_i, \vec{z}_i)$ linked to the observer. This frame is called the study frame.

The components of the velocity $\vec{V}^i(P)$ and acceleration vectors $\vec{\gamma}^i(P)$ being measured and defined in the frame $R_i(O, \vec{x}_i, \vec{y}_i, \vec{z}_i)$, we can know their components in any frame of space $R_p(O, \vec{x}_P, \vec{y}_P, \vec{z}_P)$, which we will call the projection frame.

Choosing this projection frame allows us to express the kinematic parameters with simpler mathematical expressions. It is often interesting to choose the projection frame different from the study frame to simplify and reduce calculations. The projection frame being mobile relative to the study frame, care must be taken during derivations as the unit vectors of the projection frame change direction and this must be accounted for.

III.7.3 Motion of a Frame R_k Relative to a Frame R_i Linked to the Observer

Let $R_i(O_i,\vec{x}_i,\vec{y}_i,\vec{z}_i)$ be a frame linked to the observer and $R_k(O_k,\vec{x}_k,\vec{y}_k,\vec{z}_k)$ a frame in any motion relative to the first. Any point in space can be completely located in R_k and its components deduced in R_i or conversely by knowing the motion of R_k relative to R_i. The motion of the frame R_k is completely known if:

- The position of its center O_k is completely known in R_i;
- The orientation of the axes of R_k is known relative to those of R_i.

III.7.3.1 Location of the Center O_k of the Frame R_k

The location of the center point O_k of the frame R_k is determined by the components of the vector $\overrightarrow{O_iO_k}$ linking the two centers of the frames in R_i or R_k, which results in the following relations:

$$\text{In } R_i:\ \left\{\begin{array}{l}\overrightarrow{O_iO_k}.\vec{x}_i\\ \overrightarrow{O_iO_k}.\vec{y}_i\\ \overrightarrow{O_iO_k}.\vec{z}_i\end{array}\right._{R_i}\quad \text{In } R_k:\ \left\{\begin{array}{l}\overrightarrow{O_iO_k}.\vec{x}_k\\ \overrightarrow{O_iO_k}.\vec{y}_k\\ \overrightarrow{O_iO_k}.\vec{z}_k\end{array}\right._{R_k}$$

III.7.3.2 Formula for the Mobile Basis

Let $R_i(O_i,\vec{x}_i,\vec{y}_i,\vec{z}_i)$ be a fixed frame and $R_k(O_k,\vec{x}_k,\vec{y}_k,\vec{z}_k)$ a frame mobile relative to the first. The unit vectors of the frame R_k are orthogonal to each other and have constant modules equal to 1, but they change direction in space.

$\|\vec{x}_k\| = \|\vec{y}_k\| = \|\vec{z}_k\| = 1$ and $\vec{x}_k.\vec{y}_k = 0$, $\vec{x}_k.\vec{z}_k = 0$, $\vec{y}_k.\vec{z}_k = 0$

We will determine the derivatives of these vectors in the frame R_i: $\dfrac{d^i\vec{x}_k}{dt}, \dfrac{d^i\vec{y}_k}{dt}, \dfrac{d^i\vec{z}_k}{dt}$

Let $\vec{\Omega}_k^i = \dot{\theta}(a\vec{x}_k + b\vec{y}_k + c\vec{z}_k)$ be the rotation vector of the frame $R_k(O_k,\vec{x}_k,\vec{y}_k,\vec{z}_k)$ relative to the frame $R_i(O_i,\vec{x}_i,\vec{y}_i,\vec{z}_i)$.

We then have the following relations:

$$\frac{d^i\vec{x}_k}{d\theta} \perp \vec{x}_k \Rightarrow \frac{d^i\vec{x}_k}{d\theta} \in (\vec{y}_k,\vec{z}_k);\ \text{We can write: } \frac{d^i\vec{x}_k}{d\theta} = 0.\vec{x}_k + c\vec{y}_k - b\vec{z}_k$$

$$\frac{d^i\vec{x}_k}{dt} = \frac{d^i\vec{x}_k}{d\theta}\frac{d^i\theta}{dt} = (0.\vec{x}_k + c\vec{y}_k - b\vec{z}_k)\dot{\theta} = \dot{\theta}\begin{pmatrix}a\\b\\c\end{pmatrix} \wedge \begin{pmatrix}1\\0\\0\end{pmatrix} = \vec{\Omega}_k^i \wedge \vec{x}_k$$

$\frac{d^i\vec{y}_k}{d\theta} \perp \vec{y}_k \Rightarrow \frac{d^i\vec{y}_k}{d\theta} \in (\vec{x}_k, \vec{z}_k)$; We can write: $\frac{d^i\vec{y}_k}{d\theta} = -c.\vec{x}_k + 0\vec{y}_k + a\vec{z}_k$

$$\frac{d^i\vec{y}_k}{dt} = \frac{d^i\vec{y}_k}{d\theta}\frac{d^i\theta}{dt} = (-c.\vec{x}_k + 0\vec{y}_k + a\vec{z}_k)\dot{\theta} = \dot{\theta}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \vec{\Omega}_k^i \wedge \vec{y}_k$$

$\frac{d^i\vec{z}_k}{d\theta} \perp \vec{z}_k \Rightarrow \frac{d^i\vec{z}_k}{d\theta} \in (\vec{x}_k, \vec{y}_k)$; We can write: $\frac{d^i\vec{z}_k}{d\theta} = b.\vec{x}_k - a\vec{y}_k + 0\vec{z}_k$

$$\frac{d^i\vec{z}_k}{dt} = \frac{d^i\vec{z}_k}{d\theta}\frac{d^i\theta}{dt} = (b\vec{x}_k - a\vec{y}_k + 0.\vec{z}_k)\dot{\theta} = \dot{\theta}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \vec{\Omega}_k^i \wedge \vec{z}_k$$

So we have: $\frac{d^i\vec{x}_k}{dt} = \vec{\Omega}_k^i \wedge \vec{x}_k$; $\frac{d^i\vec{y}_k}{dt} = \vec{\Omega}_k^i \wedge \vec{y}_k$; $\frac{d^i\vec{z}_k}{dt} = \vec{\Omega}_k^i \wedge \vec{z}_k$

III.7.3.3 Derivative in the Frame R_i of a Vector Expressed in a Frame R_k

The vector $\vec{V}(t)$ can be written as $\vec{V}(t) = X_k\vec{x}_k + Y_k\vec{y}_k + Z_k\vec{z}_k$ in the frame R_k.

Its derivative in the frame R_k is expressed as: $\frac{d^k\vec{V}(t)}{dt} = \dot{X}_k\vec{x}_k + \dot{Y}_k\vec{y}_k + \dot{Z}_k\vec{z}_k$

Its derivative in the frame R_i is written as:

$$\frac{d^i\vec{V}(t)}{dt} = \frac{d^k\vec{V}(t)}{dt} + X_k\vec{\Omega}_k^i\vec{x}_k + Y_k\vec{\Omega}_k^i\vec{y}_k + Z_k\vec{\Omega}_k^i\vec{z}_k$$

$$\frac{d^i\vec{V}(t)}{dt} = \frac{d^k\vec{V}(t)}{dt} + \vec{\Omega}_k^i \wedge (X_k\vec{x}_k + Y_k\vec{y}_k + Z_k\vec{z}_k = \frac{d^k\vec{V}(t)}{dt} + \vec{\Omega}_k^i \wedge \vec{V}(t)$$

Finally, we obtain: $\frac{d^i\vec{V}(t)}{dt} = \frac{d^k\vec{V}(t)}{dt} + \vec{\Omega}_k^i \wedge \vec{V}(t)$

III.7.3.4 Properties of the Vector $\vec{\Omega}_k^i$

a) The vector $\vec{\Omega}_k^i$ is antisymmetric with respect to indices i and j: $\vec{\Omega}_k^i = -\vec{\Omega}_i^k$

b) Chasles' formula: $\vec{\Omega}_k^i = \vec{\Omega}_k^j + \vec{\Omega}_j^i$ (principle of composition)

c) $\frac{d^i\vec{\Omega}_k^i}{dt} = \frac{d^k\vec{\Omega}_k^i}{dt}$ Equality of derivatives with respect to indices.

III.7.4 Transition Matrix (Type 1 Euler Angles)

Let $R_i(O_i,\vec{x}_i,\vec{y}_i,\vec{z}_i)$ be a fixed frame and $R_k(O_k,\vec{x}_k,\vec{y}_k,\vec{z}_k)$ a frame linked to the solid (S) in any motion in space. The center O_k of the frame R_k belongs to the solid $O_k \in$ (S). In the case of type 1 Euler angles, we consider that the centers O_i and O_k of the two frames are coincident: $O_i \equiv O_k$, which means that the frame R_k only undergoes rotations relative to the frame R_i. Three independent parameters are necessary to completely define the orientation of the frame R_k relative to that of R_i.

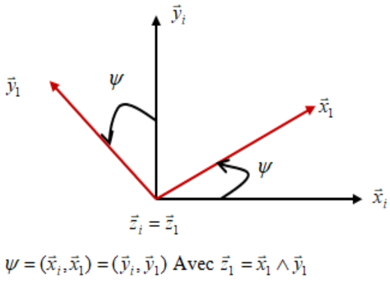
The transition from frame R_k to frame R_i is achieved by three rotations using two intermediate frames R_1 and R_2.

III.7.4.1 Transition from Frame R₁ to Frame Rᵢ: (the yaw rotation)

The rotation is performed around the axis $\vec{z}_i \equiv \vec{z}_1$.

We transition from frame $R_i(O_i,\vec{x}_i,\vec{y}_i,\vec{z}_i)$ to frame $R_1(O_1,\vec{x}_1,\vec{y}_1,\vec{z}_1)$ by rotating by an angle ψ: called the precession angle. The rotation speed is given by:

$$\vec{\Omega}_1^i = \dot{\psi}\vec{z}_i = \dot{\psi}\vec{z}_1 \text{ Because } \vec{z}_i \text{ is confused with } \vec{z}_1$$



The representation is done by plane figures from which we construct the transition matrices. Thus, we have:

$$\vec{x}_1 = \cos\psi\vec{x}_i + \sin\psi\vec{y}_i + 0.\vec{z}_i$$
$$\vec{y}_1 = -\sin\psi\vec{x}_i + \cos\psi\vec{y}_i + 0.\vec{z}_i$$
$$\vec{z}_1 = 0.\vec{x}_i + 0.\vec{y}_i + \vec{z}_i$$

These three equations can be written in matrix form, and we obtain:

$$\begin{pmatrix}\vec{x}_1\\ \vec{y}_1\\ \vec{z}_1\end{pmatrix} = \begin{pmatrix}\cos\psi & \sin\psi & 0\\ -\sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{pmatrix} . \begin{pmatrix}\vec{x}_i\\ \vec{y}_i\\ \vec{z}_i\end{pmatrix}$$

$$P_{R_1\to R_i} = \begin{pmatrix}\cos\psi & \sin\psi & 0\\ -\sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{pmatrix}$$ This is the transition matrix from frame R_1 to frame R_i.

The transition matrix from R_i to R_1 is equal to the transpose of the above matrix $P_{R_1\to R_i}$: $P_{R_i\to R_1} = P^T{}_{R_1\to R_i}$.

III.7.4.2 Transition from Frame R₂ to Frame R₁: (the pitch rotation)

The rotation is performed around the axis $\vec{x}_1 \equiv \vec{x}_2$.

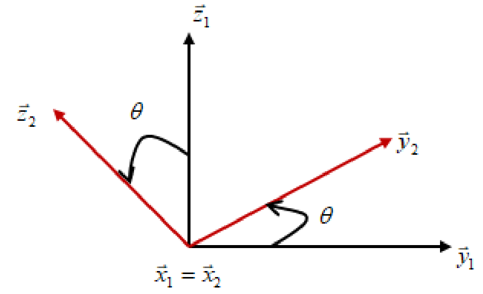
We transition from frame $R_2(O_2,\vec{x}_2,\vec{y}_2,\vec{z}_2)$ to frame $R_1(O_1,\vec{x}_1,\vec{y}_1,\vec{z}_1)$ by rotating by an angle θ: called the nutation angle. The rotation speed is given by:

$\vec{\Omega}_2^1 = \dot{\theta}\vec{x}_1 = \dot{\theta}\vec{x}_2$

$\vec{\Omega}_2^1 = \dot{\theta}\vec{x}_1 = \dot{\theta}\vec{x}_2$ Because $\vec{x}_1$ is confused with $\vec{x}_2$

Thus, we have:

$$\vec{x}_2 = \vec{x}_i + 0.\vec{y}_1 + 0.\vec{z}_1$$
$$\vec{y}_2 = 0.\vec{x}_1 + \cos\theta\vec{y}_1 + \sin\theta\vec{z}_i$$
$$\vec{z}_1 = 0.\vec{x}_1 - \sin\theta\vec{y}_1 + \cos\theta\vec{z}_i$$



$\theta = (\vec{y}_1, \vec{y}_2) = (\vec{z}_1, \vec{z}_2)$ Avec $\vec{x}_2 = \vec{y}_2 \wedge \vec{z}_2$

In matrix form we get:

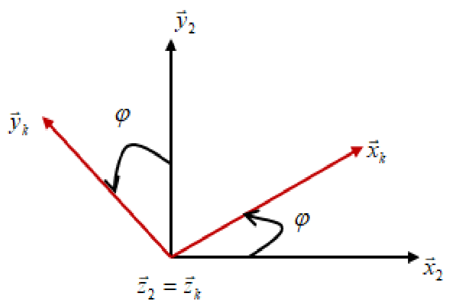
$$\begin{pmatrix} \vec{x}_1 \\ \vec{y}_1 \\ \vec{z}_1 \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} \vec{x}_i \\ \vec{y}_i \\ \vec{z}_i \end{pmatrix}$$

$$P_{R_1 \to R_i} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ This is the transition matrix from frame R_2 to frame R_1

III.7.4.3 Transition from Frame R_k to Frame R_2: (the roll rotation)

The rotation is performed around the axis $\vec{z}_2 \equiv \vec{z}_k$.

We transition from frame R_k to frame R_2 by rotating by an angle φ: called the proper rotation angle. The rotation speed is given by:



$\varphi = (\vec{x}_2, \vec{x}_k) = (\vec{y}_2, \vec{y}_k)$ Avec $\vec{z}_k = \vec{x}_k \wedge \vec{y}_k$

$\vec{\Omega}_k^2 = \dot{\varphi}\vec{z}_2 = \dot{\varphi}\vec{z}_k$ Because $\vec{z}_i$ is confused with $\vec{z}_1$

Thus, we have:

$$\vec{x}_k = \cos\varphi\vec{x}_2 + \sin\varphi\vec{y}_2 + 0.\vec{z}_2$$
$$\vec{y}_k = -\sin\varphi\vec{x}_2 + \cos\varphi\vec{y}_2 + 0.\vec{z}_2$$
$$\vec{z}_k = 0.\vec{x}_2 + 0.\vec{y}_2 + \vec{z}_2$$

In matrix form we get:

$$\begin{pmatrix} \vec{x}_k \\ \vec{y}_k \\ \vec{z}_k \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} \vec{x}_2 \\ \vec{y}_2 \\ \vec{z}_2 \end{pmatrix}$$

$$P_{R_k \to R_2} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ This is the transition matrix from frame R_k to frame R_2

The passage from the R_k reference to the R_i reference or vice versa is done by three successive rotations such that all the axes of R_k occupy positions different from that of R_i. The transition matrix from R_k to R_i is given by the product of the three successive matrices, we obtain:

$$\begin{pmatrix} \vec{x}_k \\ \vec{y}_k \\ \vec{z}_k \end{pmatrix} = \begin{pmatrix} \cos\varphi\cos\psi - \sin\varphi\cos\theta\sin\psi & \cos\varphi\sin\psi + \sin\varphi\cos\theta\cos\psi & \sin\varphi\sin\theta \\ -\sin\varphi\cos\psi - \sin\psi\cos\theta\cos\varphi & -\sin\varphi\sin\psi + \cos\varphi\cos\theta\cos\psi & \cos\varphi\sin\theta \\ \sin\theta\sin\psi & -\sin\theta\cos\psi & \cos\theta \end{pmatrix} . \begin{pmatrix} \vec{x}_i \\ \vec{y}_i \\ \vec{z}_i \end{pmatrix}$$

The transition matrix from R_i to R_k is given by the transpose of the latter.

The instantaneous rotation vector of the reference frame R_k with respect to R_i will have the vector expression:

$$\vec{\Omega}_k^i = \dot{\psi}\vec{z}_i + \dot{\theta}\vec{x}_1 + \dot{\varphi}\vec{z}_2$$

It will have a different expression depending on whether it is written in one or the other of the two markers.

In R_i, we will have: $\vec{\Omega}_k^i = \left\{ \begin{array}{c} \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ -\dot{\varphi}\sin\theta\cos\psi + \dot{\theta}\sin\psi \\ \dot{\varphi}\cos\psi + \dot{\psi} \end{array} \right._{R_i}$

In R_k, we will have: $\vec{\Omega}_k^i = \left\{ \begin{array}{c} \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi \\ \dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi \\ \dot{\varphi} + \dot{\psi}\cos\psi \end{array} \right._{R_k}$

This instantaneous rotation vector allows deducing the speed of all the solid points by knowing the speed of a single point belonging to the solid.

III.8 Fields of Velocity and Acceleration of a Solid

Consider a fixed reference frame $R_i\,(O_i, \vec{x}_i, \vec{y}_i, \vec{z}_i)$ and a solid (S_k) linked to a moving reference frame $R_k\,(O_k, \vec{x}_k, \vec{y}_k, \vec{z}_k)$ in space. For any point on the solid (S_k), we can associate its position vector, thus its velocity vector and acceleration vector.

Consider two points A_k and B_k belonging to the solid (S_k). We will seek a relationship between their velocities and their accelerations.

III.8.1 Velocity Fields

The solid (S_k) is non-deformable, so the distance $\overrightarrow{A_k B_k} = Cte$ remains constant over time in both reference frames. This vector will be expressed differently in R_i and R_k. The velocities of points A_K and B_k are different because the solid has arbitrary motion.

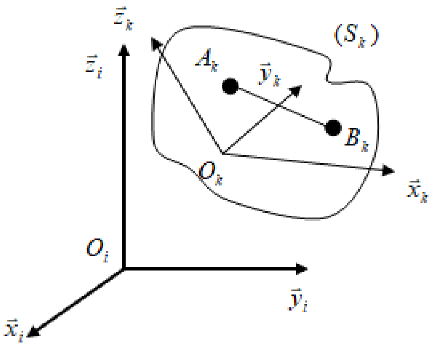


Figure III. 7: Velocity fields

In reference frame R_i: $\overrightarrow{O_i B_k} = \overrightarrow{O_i A_k} + \overrightarrow{A_k B_k} \Rightarrow \overrightarrow{A_k B_k} = \overrightarrow{O_i B_k} - \overrightarrow{O_i A_k} = Cte$

In reference frame R_k: $\overrightarrow{O_k B_k} = \overrightarrow{O_k A_k} + \overrightarrow{A_k B_k} \Rightarrow \overrightarrow{A_k B_k} = \overrightarrow{O_k B_k} - \overrightarrow{O_k A_k} = Cte$

From these two expressions, we can deduce a relationship between the velocities of the two points belonging to the solid.

The velocities of the two points with respect to the reference frame R_i are given by:

$$\vec{V}^i(A_k) = \frac{d^i \overrightarrow{O_i A_K}}{dt} \text{ and } \vec{V}^i(B_k) = \frac{d^i \overrightarrow{O_i B_K}}{dt}$$

These two expressions can be written as:

$$\vec{V}^i(A_k) = \frac{d^i \overrightarrow{O_i A_K}}{dt} = \frac{d^k \overrightarrow{O_i A_K}}{dt} + \vec{\Omega}_k^i \wedge \overrightarrow{O_i A_k} \text{ ....(1)}$$

$$\vec{V}^i(B_k) = \frac{d^i \overrightarrow{O_i B_K}}{dt} = \frac{d^k \overrightarrow{O_i B_K}}{dt} + \vec{\Omega}_k^i \wedge \overrightarrow{O_i B_k} \text{ ....(2)}$$

By subtracting the two expressions (2) - (1):

$$\vec{V}^i(B_k) - \vec{V}^i(A_k) = \frac{d^i (\overrightarrow{O_i B_K} - \overrightarrow{O_i A_K})}{dt} + \vec{\Omega}_k^i \wedge (\overrightarrow{O_i B_k} - \overrightarrow{O_i A_k})$$

We know that: $\frac{d^i (\overrightarrow{O_i B_K} - \overrightarrow{O_i A_K})}{dt} = \frac{d^i \overrightarrow{A_k B_k}}{dt} = 0$ because $\overrightarrow{O_i B_k} - \overrightarrow{O_i A_k} = \overrightarrow{A_k B_k}$

Thus, we obtain the distribution relationship of velocities in a solid: $\vec{V}^i(B_k) = \vec{V}^i(A_k) + \vec{\Omega}_k^i \wedge (\overrightarrow{A_k B_k})$

This relationship is of great importance in the kinematics and dynamics of solids. It allows us, from the velocity of one point of the solid, to deduce the velocity of all other points of the solid by knowing the rotational velocity of the associated reference frame.

Note:

a) If the rotation vector is zero $\Omega_k^i = 0$, then the solid is in pure translation motion, and all points of the solid have the same velocity: $\vec{V}^i(B_k) = \vec{V}^i(A_k)$;

b) If $\vec{V}^i(A_k) = 0$ and $\vec{V}^i(B_k) = \vec{\Omega}_k^i \wedge (\overrightarrow{A_k B_k})$, the solid is in pure rotational motion around the point $A_k \in (S_k)$;

c) The general motion of a solid can be described as a composition of a translation motion of point $A_k \in (S_k)$ at velocity $\vec{V}^i(A_k)$ and a rotational motion around point $A_k \in (S_k)$ at rotational velocity Ω_k^i.

III.8.2 Equiprojectivity of the Velocity Field of a Solid

We can demonstrate it in two different methods.

a) Previously, we showed that: $\vec{V}^i(B_k) = \vec{V}^i(A_k) + \vec{\Omega}_k^i \wedge (\overrightarrow{A_k B_k})$

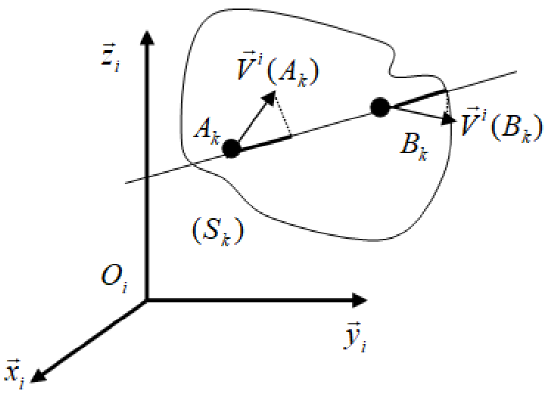


Figure III.8 : Equiprojectivity of the velocity field

By multiplying this expression by the vector $\overrightarrow{A_k B_k}$, we obtain:

$$\overrightarrow{A_k B_k}.\vec{V}^i(B_k) = \overrightarrow{A_k B_k}.\vec{V}^i(A_k) + \overrightarrow{A_k B_k}.(\vec{\Omega}_k^i \wedge \overrightarrow{A_k B_k})$$

By circular permutation of the mixed product, we can easily see that the expression:

$$\overrightarrow{A_k B_k}.(\vec{\Omega}_k^i \wedge \overrightarrow{A_k B_k}) = \vec{\Omega}_k^i.(\overrightarrow{A_k B_k} \wedge \overrightarrow{A_k B_k}) = \vec{0}$$

Thus, we obtain the equality: $\overrightarrow{A_k B_k}.\vec{V}^i(B_k) = \overrightarrow{A_k B_k}.\vec{V}^i(A_k)$

(Property of Equiprojectivity of the Velocity Field of the Solid)

b) This expression can be found another way. The solid is non-deformable, and the distance $\overrightarrow{A_k B_k}$ is constant, thus:

$$\frac{d(\overrightarrow{A_k B_k})^2}{dt} = 0$$

$$\frac{d(\overrightarrow{A_k B_k})^2}{dt} = 2\overrightarrow{A_k B_k}\frac{d\overrightarrow{A_k B_k}}{dt} = 0$$

$2\overrightarrow{A_k B_k}(\vec{V}^i(B_k) - \vec{V}^i(A_k)) = 0$ From where: $\overrightarrow{A_k B_k}.\vec{V}^i(B_k) = \overrightarrow{A_k B_k}.\vec{V}^i(A_k)$

This equiprojectivity property implies the existence of a free vector $\vec{\Omega}_k^i$ such that: $\vec{V}^i(B_k) = \vec{V}^i(A_k) + \vec{\Omega}_k^i \wedge (\overrightarrow{A_k B_k})$ which allows us to introduce the notion of kinematic screw.

III.8.3 Acceleration Fields

For each point of the solid (S_k) linked to the reference frame R_k, we deduce the acceleration from the velocity using the relation: $\vec{\gamma}^i(A_k) = \dfrac{d^i\vec{V}^i(A_k)}{dt}$

We will find a relationship linking the accelerations: $\vec{\gamma}^i(A_k)$ and $\vec{\gamma}^i(B_k)$.

We have already established a relationship between the velocities of the two points: $\vec{V}^i(B_k) = \vec{V}^i(A_k) + \vec{\Omega}_k^i \wedge (\overrightarrow{A_k B_k})$

We deduce the relationship between the accelerations by differentiating the expression of velocities.

$$\vec{\gamma}^i(B_k) = \frac{d^i\vec{V}^i(B_k)}{dt} = \frac{d^i\vec{V}^i(A_k)}{dt} + \frac{d^i\vec{\Omega}_k^i}{dt} \wedge \overrightarrow{A_k B_k} + \vec{\Omega}_k^i \wedge \frac{d^i\overrightarrow{A_k B_k}}{dt}$$

And since: $\dfrac{d^i\overrightarrow{A_k B_k}}{dt} = \dfrac{d^k\overrightarrow{A_k B_k}}{dt} + \vec{\Omega}_k^i \wedge \overrightarrow{A_k B_k} = \vec{\Omega}_k^i \wedge \overrightarrow{A_k B_k}$ because $\dfrac{d^k\overrightarrow{A_k B_k}}{dt} = \vec{0}$

Finally, we obtain the relation between the accelerations of the two points A_k and B_k of the solid: $\vec{\gamma}^i(B_k) = \vec{\gamma}^i(A_k) + \dfrac{d^i\vec{\Omega}_k^i}{dt} \wedge \overrightarrow{A_k B_k} + \vec{\Omega}_k^i \wedge (\vec{\Omega}_k^i \wedge \overrightarrow{A_k B_k})$

We observe that if the rotational velocity is constant $\vec{\Omega}_k^i = \vec{0}$, the expression becomes:

$$\vec{\gamma}^i(B_k) = \vec{\gamma}^i(A_k) + \vec{\Omega}_k^i \wedge (\vec{\Omega}_k^i \wedge \overrightarrow{A_k B_k}) = \vec{\gamma}^i(A_k) - \overrightarrow{A_k B_k}(\vec{\Omega}_k^i)^2$$

III.8.4 Kinematic Screw

The distribution formula of velocities is given by the relation: $\vec{V}^i(B_k) = \vec{V}^i(A_k) + \vec{\Omega}_k^i \wedge (\overrightarrow{A_k B_k})$

The transport formula of moments between two points A_k and B_k of the solid is expressed as: $\vec{M}(B_k) = \vec{M}(A_k) + \vec{R} \wedge \overrightarrow{A_k B_k}$

We note that there is equivalence between these two equations. The velocity vector at point B_k is the moment at point B_k of a screw, which we will denote as $[C]_{B_k}$, and the resultant is none other than the instantaneous rotation vector $\vec{\Omega}_k^i$.

The kinematic screw at point B_k (or the distribution screw of velocities) relative to the motion of the solid with respect to R_i has the reduced elements:

- Instantaneous rotation vector: $\vec{\Omega}_k^i$
- Velocity at point B_k: $\vec{V}^i(B_k)$

It will be noted in the form : $[C]_{B_k} = \begin{cases} \vec{\Omega}_k^i \\ \vec{V}^i(B_k) = \vec{V}^i(A_k) + \vec{\Omega}_k^i \wedge \overrightarrow{A_k B_k} \end{cases}$

The kinematic torso is of great interest because it completely characterizes the motion of a solid relative to the Ri mark with regard to speeds. As the reduction elements of the kinematic torso are time functions, and then the kinematic torso depends on it, so it has at every moment a different result and velocity field.

III.8.5 Instantaneous axis of rotation

The instantaneous axis of rotation is the central axis of the kinematic torso. We have shown previously that the central axis is the set of points P such that the moment of the torso at this point is parallel to the resultant. In the case of the kinematic torso, the set of these points constitutes the axis whose speeds are parallel to the vector instantaneous speed of rotation.

At each instant the motion of the solid can be considered as being the composition of a rotational motion of rotation speed $\vec{\Omega}_k^i$ around the instantaneous axis and a translation whose instantaneous direction is parallel to the rotation speed vector $\vec{\Omega}_k^i$.

Let a solid (S) be linked to a reference frame R_k in any motion relative to a reference frame R_i and the instantaneous rotation vector $\vec{\Omega}_k^i$ of the solid relative to R_i.

We consider a point A $\in$ (S). Let (π) be a normal plane $\vec{n}$ containing point A such that the rotation speed of the solid is parallel to $\vec{n} : \vec{\Omega}_k^i = \Omega_k^i\, \vec{n}$. The velocity vector of point A $\in$ (π) can be decomposed into two vectors, one in the plane (π) and the other perpendicular to (π), which gives:

$\vec{V}(A) = \vec{V}_t(A) + \vec{V}_n(A)$ with $\vec{V}_t(A) \in (\pi) et \vec{V}_n(A) \perp (\pi)$

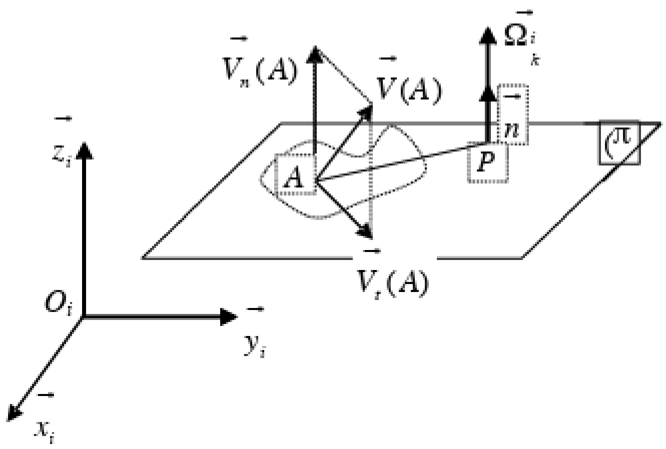


Figure III.9: Instantaneous axis of rotation

From what has been developed on the torsors, it is possible to find a P point such as: $\vec{V}_t(A) = \vec{\Omega}_k^i \wedge \overrightarrow{PA}$ then the expression of the speed of point A will be written:

$$\vec{V}(A) = \vec{V}_n(A) + \vec{\Omega}_k^i \wedge \overrightarrow{PA}$$

Whatever Q$\in$ (π) we can write by the transport formula:

$$\vec{V}(Q) = \vec{V}(A) + \vec{\Omega}_k^i \wedge \overrightarrow{AQ} = \vec{V}_n(A) + \vec{\Omega}_k^i \wedge \overrightarrow{PA} + \vec{\Omega}_k^i \wedge \overrightarrow{AQ} = \vec{V}_n(A) + \vec{\Omega}_k^i \wedge \overrightarrow{PQ}$$

$$\vec{V}(Q) = \vec{V}_n(A) + \vec{\Omega}_k^i \wedge \overrightarrow{PQ}$$

We can conclude that the velocity vector of the point Q $\in$ (π) is written:

$$\vec{V}(Q) = \vec{V}_t(Q) + \vec{V}_n(Q)$$

With $\vec{V}_t(Q) = \vec{\Omega}_k^i \wedge \overrightarrow{PQ}$ and $\vec{V}_n(Q) = \vec{V}_n(A)$

It can be seen that the velocity component, normal to the plane (π) is the same for all points of the solid. We finally get whatever P and Q:

$$\vec{V}(Q) = \vec{V}_n(A) + \vec{\Omega}_k^i \wedge \overrightarrow{PQ}$$

The motion of the solid in this case decomposes at each moment into a motion of translation in the plane and a motion of rotation around an axis passing through the point P and parallel to the unitary vector $\vec{n}$.

The axis thus defined by the point P and the unit $\vec{n} // \vec{\Omega}_k^i$ constitutes the instantaneous axis of rotation of the solid with respect to the frame R_i.

We know that the central axis of a torso is the place of the points P where the moment is minimum or zero. In the case of a kinematic torsor, the instantaneous speed is zero at all points of the central axis. We deduce that if the speed is zero, in two distinct points of a solid, then the axis joining the two points is necessarily an axis of rotation so a central axis of the kinematic torsor.

III.9 Laws of Composition of motion

III.9.1 Law of Composition of Velocities

Consider $R_i\,(O_i, \vec{x}_i, \vec{y}_i, \vec{z}_i)$ a fixed reference frame and $R_k\,(O_k, \vec{x}_k, \vec{y}_k, \vec{z}_k)$ a reference frame moving arbitrarily with respect to the fixed frame. We consider a solid (S_k) whose motion is known in the relative frame $R_k\,(O_k, \vec{x}_k, \vec{y}_k, \vec{z}_k)$.

Let P be a point on the solid. We can write at any moment:

$$\overrightarrow{O_iP} = \overrightarrow{O_iO_k} + \overrightarrow{O_kP}$$

The velocity of point P in the frame R_i is given by the derivative of the vector $\overrightarrow{O_iP}$ in the same frame.

$$\vec{V}^i(P) = \frac{d^i\overrightarrow{O_iP}}{dt} = \frac{d^i\overrightarrow{O_iO_k}}{dt} + \frac{d^i\overrightarrow{O_kP}}{dt}$$

Developing the two terms of velocity gives:

$\dfrac{d^i\overrightarrow{O_iO_k}}{dt} = \vec{V}^i(O_k)$: velocity of the center of the frame R_k with respect to the frame R_i.

$$\frac{d^i\overrightarrow{O_kP}}{dt} = \frac{d^k\overrightarrow{O_kP}}{dt} + \vec{\Omega}_k^i \wedge \overrightarrow{O_kP} = \vec{V}^k(P) + \vec{\Omega}_k^i \wedge \overrightarrow{O_kP}$$

Finally, the velocity of point P in the frame R_i is written as: $\vec{V}^i(P) = \vec{V}^k(P) + (\vec{V}^i(O_k) + \vec{\Omega}_k^i \wedge \overrightarrow{O_kP})$

This can also be written in the form: $\vec{V}^i(P) = \vec{V}^k(P) + \vec{V}^i(O_k) + \vec{V}_k^i(P)$

where:

$\vec{V}^i(P)$: Absolute velocity of point P for an observer in R_i

$\vec{V}^k(P)$: Relative velocity of point P with respect to R_k moving with respect to R_i

$\vec{V}_k^i(P)$: transport velocity of point P if it were stationary in R_k.

Note:

$\vec{V}_k^i(P) = -\vec{V}_i^k(P)$: Antisymmetric with respect to the indices, and thus to the reference frames.

$\vec{V}_k^i(P) = \vec{V}_k^j(P) + \vec{V}_j^i(P)$

III.9.2 Law of Composition of Accelerations

The absolute acceleration $\gamma^i(P)$ of point P is derived from the absolute velocity:
$\vec{\gamma}^i(P) = \dfrac{d^i\vec{V}^i(P)}{dt} = \dfrac{d^i\vec{V}^k(P)}{dt} + \dfrac{d^i\vec{V}^i(O_k)}{dt} + \dfrac{d^i(\vec{\Omega}_k^i \wedge \overrightarrow{O_kP}}{dt}$

Developing each of the three terms:

1. $\dfrac{d^i\vec{V}^i(P)}{dt} = \dfrac{d^k\vec{V}^k(P)}{dt} + \vec{\Omega}_k^i \wedge \vec{V}^k(P) = \vec{\gamma}^k(P) + \vec{\Omega}_k^i \wedge \vec{V}^k(P)$;

2. $\dfrac{d^i\vec{V}^i(O_k)}{dt} = \vec{\gamma}^i(O_k)$;

3. $\dfrac{d^i(\vec{\Omega}_k^i \wedge \overrightarrow{O_kP})}{dt} = \dfrac{d^i\vec{\Omega}_k^i}{dt} \wedge \overrightarrow{O_kP} + \vec{\Omega}_k^i \wedge \dfrac{d^i\overrightarrow{O_kP}}{dt}$

$$\frac{d^i(\vec{\Omega}_k^i \wedge \overrightarrow{O_kP})}{dt} = \frac{d^i\vec{\Omega}_k^i}{dt} \wedge \overrightarrow{O_kP} + \vec{\Omega}_k^i \wedge \frac{d^i\overrightarrow{O_kP}}{dt} = \frac{d^i\vec{\Omega}_k^i}{dt} \wedge \overrightarrow{O_kP} + \vec{\Omega}_k^i \wedge (\frac{d^k\overrightarrow{O_kP}}{dt} + \vec{\Omega}_k^i \wedge \overrightarrow{O_kP})$$
$$= \frac{d^i\vec{\Omega}_k^i}{dt} \wedge \overrightarrow{O_kP} + \vec{\Omega}_k^i \wedge (\vec{V}^k(P) + \vec{\Omega}_k^i \wedge \overrightarrow{O_kP})$$

Summing the three terms gives:

$$\vec{\gamma}^i(P) = \vec{\gamma}^k(P) + \vec{\Omega}_k^i \wedge \vec{V}^k(P) + \vec{\gamma}^i(O_k) = \frac{d^i\vec{\Omega}_k^i}{dt} \wedge \overrightarrow{O_kP} + \vec{\Omega}_k^i \wedge (\vec{V}^k(P) + \vec{\Omega}_k^i \wedge \overrightarrow{O_kP})$$

$$\vec{\gamma}^i(P) = \vec{\gamma}^k(P) + \left(\vec{\gamma}^i(O_k) + \frac{d^i\vec{\Omega}_k^i}{dt} \wedge \overrightarrow{O_kP} + \vec{\Omega}_k^i \wedge (\vec{\Omega}_k^i \wedge \overrightarrow{O_kP})\right) + 2\vec{\Omega}_k^i \wedge \vec{V}^k(P)$$

This expression can be written in a reduced form:

$$\vec{\gamma}^i(P) = \vec{\gamma}^k(P) + \vec{\gamma}_k^i(P) + \vec{\gamma}_c(P)$$

where:

$\vec{\gamma}^i(P)$: Absolute acceleration of point P (with respect to fixed R_i)

$\vec{\gamma}^k(P)$: Relative acceleration of point P (with respect to frame R_k)

$\vec{\gamma}_k^i(P) = \vec{\gamma}^i(O_k) + \dfrac{d^i\vec{\Omega}_k^i}{dt} \wedge \overrightarrow{O_kP} + \vec{\Omega}_k^i \wedge (\vec{\Omega}_k^i \wedge \overrightarrow{O_kP})$: Transport acceleration of frame R_k

$\vec{\gamma}_c(P) = 2\vec{\Omega}_k^i \wedge \vec{V}^k(P)$: Coriolis acceleration (complementary acceleration).

The Coriolis acceleration is a composition between the rotational velocity $\vec{\Omega}_k^i$ of the frame R_k with respect to the frame R_i and the relative velocity $\vec{V}^k(P)$ of point P.

The Coriolis acceleration of point P is zero if and only if:

- The rotational velocity of the relative frame with respect to the absolute frame is zero: $\vec{\Omega}_k^i = \vec{0}$;
- The relative velocity of point P is zero: $\vec{V}^k(P) = \vec{0}$;
- The rotational velocity is collinear with the relative velocity: $\vec{\Omega}_k^i // \vec{V}^k(P)$.

III.10 Fundamental Particular Motions

III.10.1 Pure Translation motion

A solid (S_k) linked to a frame $R_k(O_k, \vec{x}_k, \vec{y}_k, \vec{z}_k)$ is said to be in pure translation motion with respect to a frame $R_i(O_i, \vec{x}_i, \vec{y}_i, \vec{z}_i)$ if the axes of $R_k(O_k, \vec{x}_k, \vec{y}_k, \vec{z}_k)$ maintain a fixed direction with respect to those of $R_i(O_i, \vec{x}_i, \vec{y}_i, \vec{z}_i)$ over time.

All points of the solid have the same velocity and the same acceleration as point P∈ (S_k).

The rotational velocity of the solid is zero with respect to R_i.

We can write: $\vec{V}^i(P) = \vec{V}^i(O_k)$ and $\vec{\Omega}_k^i \wedge \overrightarrow{O_k P} = \vec{0}$

Since $\overrightarrow{O_k P} \neq \vec{0}$, then $\vec{\Omega}_k^i = \vec{0}$.

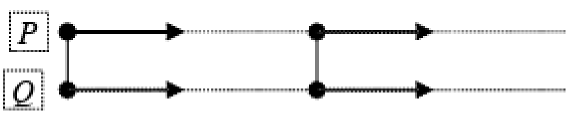
In this case, the velocity field is a uniform field.

The kinematic screw describing pure translation motion is a zero-couple screw with a resultant that is zero but a non-zero moment.

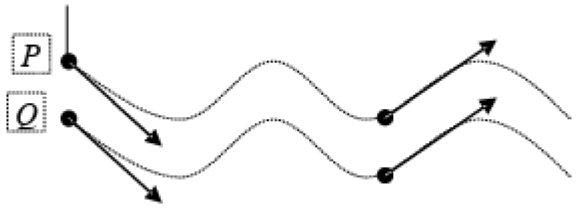
$$[C]_{k/i} = \begin{cases} \vec{\Omega}_k^i = \vec{0} \\ \vec{V}^i(P) = \vec{V}^i(Q) \neq \vec{0} \end{cases}$$

Since all points of the solid have the same velocity at each moment, the points describe parallel trajectories. Three types of trajectories can be described: Let P and Q be two points of the solid:

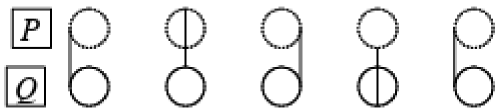
If the trajectories of the solid's points are rectilinear, it is called rectilinear translation. If their respective velocities are constant over time, we have uniform rectilinear translation.



Curvilinear translation trajectory: points P and Q have parallel and equal velocities.



Circular translation trajectory: points P and Q describe circles of the same radius at the same velocity.



III.10.2 Pure Rotation motion around an Axis of the Solid

III.10.2.1 Velocity of a Point P on the Solid

A solid (S_k) linked to a frame $R_k(O_k, \vec{x}_k, \vec{y}_k, \vec{z}_k)$ is said to be in pure rotation motion with respect to a frame $R_i(O_i, \vec{x}_i, \vec{y}_i, \vec{z}_i)$ if an axis of $R_k(O_k, \vec{x}_k, \vec{y}_k, \vec{z}_k)$ remains fixed at all times and permanently in the frame $R_i(O_i, \vec{x}_i, \vec{y}_i, \vec{z}_i)$. Thus, we have two distinct points O_k and I on the solid (S_k) that remain fixed in the frame $R_i(O_i, \vec{x}_i, \vec{y}_i, \vec{z}_i)$ during the rotation motion.

The frame $R_k(O_k, \vec{x}_k, \vec{y}_k, \vec{z}_k)$ is in pure rotation with respect to the frame $R_i(O_i, \vec{x}_i, \vec{y}_i, \vec{z}_i)$ at an angular velocity given by: $\vec{\Omega}_k^i = \dot{\psi}.\vec{z}_i = \dot{\psi}.\vec{z}_k$ et $\vec{V}^i(O_k) = \vec{0}$

Let P be any point on the solid not belonging to the rotation axis such that: $\overrightarrow{IP} = r\vec{x}_k$

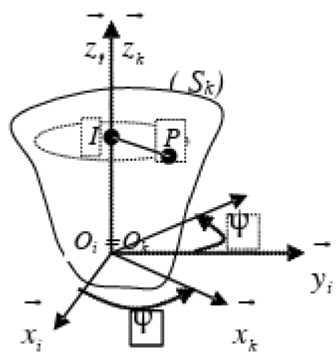


Figure III.10: Velocity of a Point P on the Solid

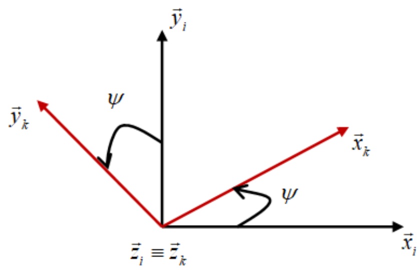
Whatever : $I \in \vec{z}_i$ et $\vec{z}_k$ We can write: $\vec{V}^i(I) = \vec{V}^i(O_k) + \vec{\Omega}_k^i \wedge \overrightarrow{O_k I}$

Thus, we get: $\vec{\Omega}_k^i // \overrightarrow{O_k I} \Rightarrow \vec{\Omega}_k^i \wedge \overrightarrow{O_k I} = \vec{0}$ where $\vec{V}^i(I) = \vec{V}^i(O_k) = \vec{0}$

I and P are two points of the solid, we can then write:

$$\vec{V}^i(P) = \vec{V}^i(I) + \vec{\Omega}_k^i \wedge \overrightarrow{IP} = \vec{\Omega}_k^i \wedge \overrightarrow{IP} \Rightarrow \vec{V}^i(P) = \vec{\Omega}_k^i \wedge \overrightarrow{IP}$$

We replace $\vec{\Omega}_k^i$ and $\overrightarrow{IP}$ by their expressions, the speed of the point P becomes:



$$\vec{V}^i(P) = \vec{\Omega}_k^i \wedge \overrightarrow{IP} = \dot{\psi}\vec{z}_k \wedge r.\vec{x}_k = r\dot{\psi}\vec{y}_k$$

In pure rotation motion, the velocity screw is equivalent to the sliding screw defined by: $[C]_{k/i} = \begin{cases} \vec{\Omega}_k^i \neq \vec{0} \\ \vec{V}^i(I) = \vec{0} \end{cases}$ with $I \in \vec{z}_i$ and $\vec{z}_k$

III.10.2.2 Acceleration of a Point P on the Solid

We previously found the velocity of point P given by: $\vec{V}^i(P) = \vec{\Omega}_k^i \wedge \overrightarrow{IP}$

By deriving this expression, we get:

$\vec{\gamma}(P) = \dfrac{d^i\vec{V}^i(P)}{dt} = \dfrac{d^i\vec{\Omega}_k^i}{dt} \wedge \overrightarrow{IP} + \vec{\Omega}_k^i \wedge \dfrac{d^i\overrightarrow{IP}}{dt}$ yet we have $\dfrac{d^i\overrightarrow{IP}}{dt} = \dfrac{d^k\overrightarrow{IP}}{dt} + \vec{\Omega}_k^i \wedge \overrightarrow{IP}$ such as $\overrightarrow{IP} = cst$ in the reference R_k then $\dfrac{d^k\overrightarrow{IP}}{dt} = \vec{0}$ Which give $\dfrac{d^i\overrightarrow{IP}}{dt} = \vec{\Omega}_k^i \wedge \overrightarrow{IP}$

Expanding this expression, we obtain: $\vec{\gamma}(P) = \dfrac{d^i\vec{\Omega}_k^i}{dt} \wedge \overrightarrow{IP} + \vec{\Omega}_k^i \wedge (\vec{\Omega}_k^i \wedge \overrightarrow{IP})$

But we have: $\vec{\Omega}_k^i \perp \overrightarrow{IP} \Rightarrow \vec{\Omega}_k^i.\overrightarrow{IP} = 0$ et $\vec{\Omega}_k^i.\vec{\Omega}_k^i = \vec{\Omega}_k^{i\,2}$

Finally, the acceleration expression becomes:

$$\vec{\gamma}(P) = \quad -\overrightarrow{IP}.\vec{\Omega}_k^{i\,2} \quad + \quad \frac{d^i\vec{\Omega}_k^i}{dt} \wedge \overrightarrow{IP}$$

where:

- $-\overrightarrow{IP}.\vec{\Omega}_k^{i\,2}$: Normal acceleration along the direction.
- $\dfrac{d^i\vec{\Omega}_k^i}{dt} \wedge \overrightarrow{IP}$: Tangential acceleration at point P.

Replacing $\vec{\Omega}_k^i = \dot{\psi}.\vec{z}_k$, $\overrightarrow{IP} = r\vec{x}_k$ and $\dfrac{d^i\vec{\Omega}_k^i}{dt} = \ddot{\psi}\vec{z}_k$ by their respective expressions:

$$\vec{\gamma}^i(p) = -r\dot{\psi}^2.\vec{x}_k + r\ddot{\psi}\vec{y}_k = \vec{\gamma}_n(P) + \vec{\gamma}_t(P)$$

Expressions of speed and acceleration can be easily expressed in the $R_i(O_i, \vec{x}_i, \vec{y}_i, \vec{z}_i)$ frame by determining the matrix of passage from the R_i frame to the R_k: $P_{R_k \to R_i}$.

$$\vec{x}_k = \cos\psi\vec{x}_i + \sin\psi\vec{y}_i + 0.\vec{z}_i$$
$$\vec{y}_k = -\sin\psi\vec{x}_i + \cos\psi\vec{y}_i + 0.\vec{z}_i$$
$$\vec{z}_k = 0.\vec{x}_i + 0.\vec{y}_i + \vec{z}_i$$

Where $P_{R_k \to R_i} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Speed and acceleration will be expressed in R_i:

$$\vec{V}^i(P) = r\dot{\psi}\vec{y}_k = r\dot{\psi}(-\sin\psi\vec{x}_i + \cos\psi\vec{y}_i) = -r\dot{\psi}\sin\psi\vec{x}_i + r\dot{\psi}\cos\psi\vec{y}_i$$

$$\vec{\gamma}^i(P) = r\dot{\psi}^2\vec{x}_k + r\ddot{\psi}\vec{y}_k = -r\dot{\psi}^2(\cos\psi\vec{x}_i + \sin\psi\vec{y}_i) + r\ddot{\psi}(-\sin\psi\vec{x}_i + \cos\psi\vec{y}_i)$$

$$\vec{\gamma}^i(P) = -r(\dot{\psi}^2\cos\psi + \ddot{\psi}\sin\psi)\vec{x}_i + r(-\dot{\psi}^2\sin\psi + \ddot{\psi}\cos\psi)\vec{y}_i$$

III.10.3 Helical motion (Rotation + Translation)

A solid (S_k) linked to a frame $R_k(O_k, \vec{x}_k, \vec{y}_k, \vec{z}_k)$ describes helical motion with respect to a fixed frame $R_i(O_i, \vec{x}_i, \vec{y}_i, \vec{z}_i)$ if:

An axis of the frame $R_k(O_k, \vec{x}_k, \vec{y}_k, \vec{z}_k)$ remains coincident at all times with an axis of the frame $R_i(O_i, \vec{x}_i, \vec{y}_i, \vec{z}_i)$.

The coordinate of point O_k, the center of the frame $R_k(O_k, \vec{x}_k, \vec{y}_k, \vec{z}_k)$, along the coincident axis is proportional to the rotation angle of frame $R_k(O_k, \vec{x}_k, \vec{y}_k, \vec{z}_k)$ with respect to the frame $R_i(O_i, \vec{x}_i, \vec{y}_i, \vec{z}_i)$ during the rotation motion.

Thus, we have: $\overrightarrow{O_iO_k} = \lambda\psi(t)\vec{z}_i = \lambda\psi(t)\vec{z}_k$

The scalar λ represents the pitch of the helical motion along the coincident axis.

We have two superimposed motion:

- A translation motion along the common axis $\vec{z}_i \equiv \vec{z}_k$.
- A rotation motion around the same axis $\vec{z}_i \equiv \vec{z}_k$.

Application Exercises

Exercise 01:

A material point moves along a trajectory described by the following parametric equations: $\begin{cases} x = t \\ y = 2t^2 \\ z = 0 \end{cases}$

Determine:

1. The unit tangent vector $\vec{\tau}$ to the trajectory;
2. The radius of curvature ρ;
3. The normal $\vec{n}$ to the trajectory;
4. The binormal $\vec{b}$.

Solution 01:

1. **Unit Tangent Vector $\vec{\tau}$ to the Trajectory**

The unit tangent vector $\vec{\tau}$ has the same direction and sense as the velocity vector $\vec{\tau} = \frac{\vec{v}}{|\vec{v}|}$ **.**

The velocity is written as: $\vec{v} = \begin{cases} v_x = 1 \\ v_y = 4t \Rightarrow \vec{v} = \vec{i} + 4t\ \vec{j} \\ v_z = 0 \end{cases}$ and $\vec{\gamma} = \begin{cases} \gamma_x = 0 \\ \gamma_y = 4 \Rightarrow \gamma = 4 \\ \gamma_z = 0 \end{cases}$

And $|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{1+16t^2}$

Thus: T $\vec{\tau} = \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{i} + 4\vec{j}}{\sqrt{1+16t^2}} = \frac{1}{\sqrt{1+16t^2}}\vec{i} + \frac{4t}{\sqrt{1+16t^2}}\vec{j}$

2. **Radius of Curvature ρ**

In the Frenet frame, the acceleration of the material point is written as: $\vec{\gamma} = \vec{\gamma}_N + \vec{\gamma}_t$

Where $\vec{\gamma}_N$ and $\vec{\gamma}_t$ are the tangential and normal accelerations, respectively.

We know that: $\vec{\gamma}_N = \frac{v^2}{\rho}$ Calculating $\vec{\gamma}_N$,

Such as $\vec{\gamma}_t = \frac{dv}{dt} = \frac{1}{2}32t(1+16t^2)^{\frac{1}{2}-1} = \frac{16t}{\sqrt{1+16t^2}}$ and that $\gamma^2 = \gamma^2{}_N + \gamma^2{}_t$

We find: $\gamma^2{}_N = \gamma^2 - \gamma^2{}_t = 16 - \frac{16t^2}{1+16t^2} = \frac{16}{1+16t^2} \Rightarrow \gamma_N = \frac{16}{\sqrt{1+16t^2}}$

$$\rho = \frac{v^2}{\gamma_N} = \frac{1+16t^2}{\frac{4}{\sqrt{1+16t^2}}} = \frac{(1+16t^2)^{\frac{3}{2}}}{4}$$

3. **Normal to the Trajectory** $\vec{n}$

Let s be the arc length. The normal to the trajectory is given by: $\vec{n} = \frac{d\vec{\tau}}{d\theta} = \frac{d\vec{\tau}}{ds}\frac{ds}{d\theta} = \rho\frac{d\vec{\tau}}{ds} = \rho\frac{d\vec{\tau}}{dt}\frac{dt}{ds} = \frac{\rho}{v}\frac{d\vec{\tau}}{dt}$ **;** $v = \frac{ds}{dt}$

Since: $\vec{n} = \frac{\rho}{v}\left(\frac{4\vec{j}(1+16t^2)^{\frac{1}{2}} - (\vec{i}+4\vec{j})16(1+16t^2)^{-\frac{1}{2}}}{1+16t^2}\right) = \frac{\rho}{v}\frac{4(-4\vec{i}+\vec{j})}{(1+16t^2)^{\frac{3}{2}}} = \frac{(1-16t^2)^{\frac{3}{2}}}{4(1+16t^2)^{\frac{1}{2}}}\frac{4(-4\vec{i}+\vec{j})}{(1+16t^2)^{\frac{3}{2}}}$

$$\vec{n} = \frac{-4t}{\sqrt{1+16t^2}}\vec{i} + \frac{1}{\sqrt{1+16t^2}}\vec{j}$$

4. **Binormal**

It is a unit vector perpendicular to both the tangent and normal vectors: $\vec{b} = \vec{\tau} \wedge \vec{n}$

$$\vec{b} = \begin{pmatrix} \frac{1}{1+16t^2} \\ \frac{4t}{1+16t^2} \\ 0 \end{pmatrix} \wedge \vec{n}\begin{pmatrix} \frac{-4t}{1+16t^2} \\ \frac{1}{1+16t^2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \; ; \; \vec{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

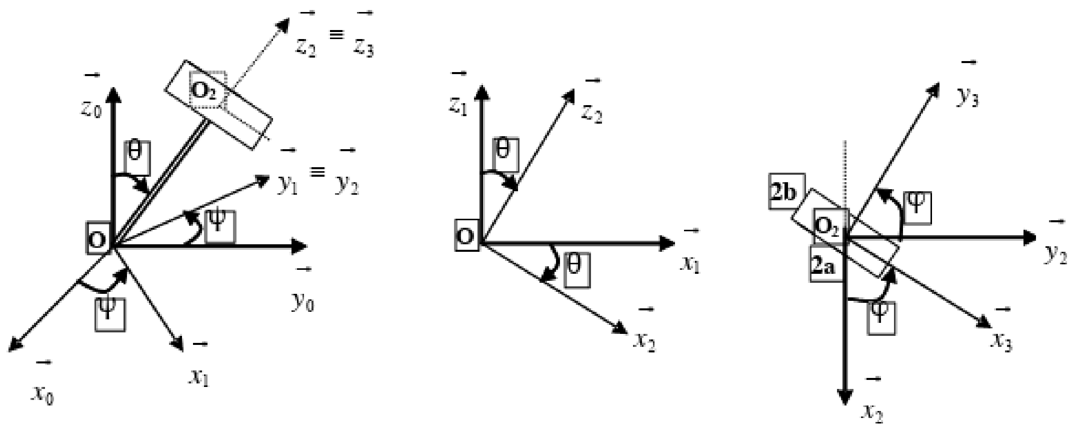
Exercise 02:

Consider the mechanical system composed of a rod O_2 of length L and a rectangular plate of dimensions 2a and 2b hinged at O_2 with the rod (see figure). R_0 being the fixed frame; R_1 rotating by Ψ around the axis $\vec{z}_0$. The plate rotates around the rod at an angular velocity $\dot{\varphi}$.

Given: $\dot{\psi}$ = Cte ; $\dot{\theta}$ = Cte ; $\dot{\varphi}$ = Cte

Determine:

1. The transformation matrices from R_1 to R_2 and from R_3 to R_2;
2. The instantaneous rotation vector of R_3 relative to R_0 expressed in R_2;
3. The velocity $\vec{V}^0(O_2)$ expressed in frame R_2 by differentiation;
4. The velocity $\vec{V}^0(A)$ with respect to R_0 expressed in R_2 by the solid's kinematics;
5. The acceleration expressed $\vec{\gamma}^0(O_2)$ in frame R_2 by differentiation and by the solid's kinematics.



Solution 02:

The rod: $OO_2=L$; The plate: Length 2a, Width 2b

$R_0(O,\vec{x}_0,\vec{y}_0,\vec{z}_0)$: Fixed frame;

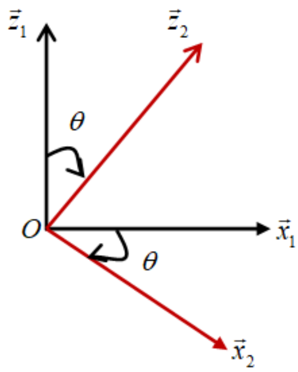
$R_1(O,\vec{x}_1,\vec{y}_1,\vec{z}_1)$: Frame rotating around the axis $\vec{z}_0$ relative to R_0;

$R_2(O,\vec{x}_2,\vec{y}_2,\vec{z}_2)$: Frame attached to the rod rotating around the axis $\vec{y}_1$ relative to R_1;

$R_3(O,\vec{x}_3,\vec{y}_3,\vec{z}_3)$: Frame attached to the plate rotating around the axis $\vec{z}_2$ relative to R_2;

Given: $\dot{\psi}=$ Cte ; $\dot{\theta}=$ Cte ; $\dot{\varphi}=$ Cte

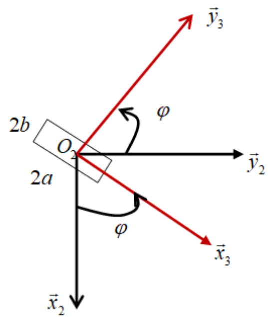
1.Transformation Matrices



Transformation matrix from R_2 to R_1:

$$\begin{pmatrix}\vec{x}_1\\ \vec{x}_2\\ \vec{x}_3\end{pmatrix}=\begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}\begin{pmatrix}\vec{x}_2\\ \vec{y}_2\\ \vec{z}_2\end{pmatrix}$$

$P_{R_1\to R_2}$



Transformation matrix from R_3 to R_2:

$$\begin{pmatrix}\vec{x}_3\\ \vec{x}_3\\ \vec{x}_3\end{pmatrix}=\begin{pmatrix}\cos\varphi & \sin\varphi & 0\\ -\sin\kappa & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\vec{x}_2\\ \vec{y}_2\\ \vec{z}_2\end{pmatrix}$$

$P_{R_3\to R_2}$

2. Instantaneous Rotation Vector of R_3 Relative to R_0 Expressed in R_2

According to Chasles' theorem, we can write:

$$\vec{\Omega}_3^0 = \vec{\Omega}_3^2 + \vec{\Omega}_2^1 + \vec{\Omega}_1^0 = \dot{\varphi}.\vec{z}_2 + \dot{\theta}.\vec{y}_2 + \dot{\psi}.\vec{z}_1$$

Expressing the unit vector $\vec{z}_1$ in frame R2, we get: $\vec{z}_1 = -\sin\theta\,\vec{x}_2 + \cos\theta\,\vec{z}_2$

$$\vec{\Omega}_3^0 = \dot{\varphi}.\vec{z}_2 + \dot{\theta}.\vec{y}_2 + \dot{\psi}(-\sin\theta\,\vec{x}_2 + \cos\theta\,\vec{z}_2) = -\dot{\psi}\sin\theta\,\vec{x}_2 + \dot{\theta}.\vec{y}_2 + (\dot{\varphi} + \dot{\psi}\cos\theta)\vec{z}_2$$

$$\vec{\Omega}_3^0 = \left\{\begin{array}{c} -\dot{\psi}\sin\theta \\ \dot{\theta} \\ \dot{\varphi} + \dot{\psi}\cos\theta \end{array}\right._{R_2}$$

3. $\vec{V}^0(O_2)$ Velocity Expressed in Frame R2 by Differentiation

By differentiation: $\vec{V}^0(O_2) = \dfrac{d^0\overrightarrow{OO_2}}{dt} = \dfrac{d^2\overrightarrow{OO_2}}{dt} + \vec{\Omega}_2^0 \wedge \overrightarrow{OO_2}$

$$\overrightarrow{OO_2} = \left\{\begin{array}{l} 0 \\ 0 \\ L \end{array}\right._{R_2} \Rightarrow \frac{d^2\overrightarrow{OO_2}}{dt} = \vec{0}\ ;\ \text{and}\ \vec{\Omega}_2^0 = \vec{\Omega}_2^1 + \vec{\Omega}_1^0 = \dot{\theta}.\vec{y}_2 + \dot{\psi}.\vec{z}_1 = \left\{\begin{array}{c} -\dot{\psi}\sin\theta \\ \dot{\theta} \\ \psi\cos\theta \end{array}\right._{R_2}$$

$$\vec{V}^0(O_2) = \left\{\begin{array}{c} -\dot{\psi}\sin\theta \\ \dot{\theta} \\ \psi\cos\theta \end{array}\right._{R_2} \wedge \left\{\begin{array}{l} 0 \\ 0 \\ L \end{array}\right._{R_2} = \left\{\begin{array}{c} L\dot{\theta} \\ L\psi\sin\theta \\ 0 \end{array}\right._{R_2}$$

4. Velocity of Point A with Respect to R0 Expressed in Frame R2

By the solid's kinematics, we write: $\vec{V}^0(A) = \vec{V}^0(O_2) + \vec{\Omega}_3^0 \wedge \overrightarrow{O_2A}$

Point A is in frame R3 with coordinates: $\overrightarrow{O_2A} = \left\{\begin{array}{l} a \\ 0 \\ 0 \end{array}\right._{R_3} = \left\{\begin{array}{c} a\cos\varphi \\ a\sin\varphi \\ 0 \end{array}\right._{R_2}$

Where $\vec{V}^0(A) = \left\{\begin{array}{c} L\dot{\theta} \\ L\psi\sin\theta \\ 0 \end{array}\right._{R_2} + \left\{\begin{array}{c} -\dot{\psi}\sin\theta \\ \dot{\theta} \\ \dot{\varphi} + \dot{\psi}\cos\theta \end{array}\right._{R_2} \wedge \left\{\begin{array}{c} a\cos\varphi \\ a\sin\varphi \\ 0 \end{array}\right._{R_2}$

$$\vec{V}^0(A) = \left\{\begin{array}{c} L\dot{\theta} - a\sin\varphi(\dot{\varphi} + \dot{\psi}\cos\theta) \\ L\dot{\psi}\sin\theta + a\cos\varphi(\dot{\varphi} + \dot{\psi}\cos\theta) \\ -a(\dot{\psi}\sin\theta\sin\theta + \dot{\theta}\cos\varphi) \end{array}\right._{R_2}$$

5. Acceleration by Differentiation and by the Solid's Kinematics in Frame R2R_2R2

5.1. **By Differentiation**

We know: θ$\dot{\psi}$ = Cte ; $\dot{\theta}$ = Cte ; $\dot{\varphi}$ = Cte.

$$\vec{\gamma}^0(O_2) = \frac{d^0\vec{V}^0(O_2)}{dt} = \frac{d^2\vec{V}^0(O_2)}{dt} + \vec{\Omega}_2^0 \wedge \vec{V}^0(O_2)$$

This gives:

$$\vec{\gamma}^0(O_2) = \begin{cases} 0 \\ L\dot{\psi}\dot{\theta}\cos\theta + \\ 0 \end{cases}_{R_2} \begin{cases} -\dot{\psi}\sin\theta \\ \dot{\theta} \\ \dot{\varphi}+\dot{\psi}\cos\theta \end{cases}_{R_2} \wedge \begin{cases} L\dot{\theta} \\ L\dot{\psi}\sin\theta = \\ 0 \end{cases}_{R_2} \begin{cases} -L\dot{\psi}^2\sin\theta\cos\theta \\ 2L\dot{\psi}\dot{\theta}\cos\theta \\ -L\dot{\theta}^2 - L\dot{\psi}^2\sin^2\theta \end{cases}_{R_2}$$

5.2. **By the Solid's Kinematics**

$$\vec{\gamma}^0(O_2) = \vec{\gamma}^0(O) + \frac{d^0\vec{\Omega}_2^0}{dt} \wedge \overrightarrow{OO_2} + \vec{\Omega}_2^0 \wedge (\vec{\Omega}_2^0 \wedge \overrightarrow{OO_2})$$

Points O and O_2belong to the rod; their velocities and accelerations are zero in the frame R_2 attached to the rod:

$\vec{\gamma}^0(O) = \vec{0}$ Because the point O is fixed in the rod

$$\frac{d^0\vec{\Omega}_2^0}{dt} \wedge \overrightarrow{OO_2} = \begin{cases} -\dot{\psi}\dot{\theta}\cos\theta \\ 0 \\ -\dot{\psi}\dot{\theta}\sin\theta \end{cases}_{R_2} \wedge \begin{cases} 0 \\ 0 = \\ L \end{cases}_{R_2} \begin{cases} 0 \\ L\dot{\psi}\dot{\theta}\cos\theta \\ 0 \end{cases}_{R_2}$$

$$\vec{\Omega}_2^0 \wedge (\vec{\Omega}_2^0 \wedge \overrightarrow{OO_2}) = \begin{cases} -L\dot{\psi}^2\sin\theta\cos\theta \\ L\dot{\psi}\dot{\theta}\cos\theta \\ -L\dot{\theta}^2 - L\dot{\psi}^2\sin^2\theta \end{cases}_{R_2}$$

Summing these three expressions gives:

$$\vec{\gamma}^0(O_2) = \begin{cases} 0 \\ L\dot{\psi}\dot{\theta}\cos\theta + \\ 0 \end{cases}_{R_2} \begin{cases} -L\dot{\psi}^2\sin\theta\cos\theta \\ L\dot{\psi}\dot{\theta}\cos\theta \\ -L\dot{\theta}^2 - L\dot{\psi}^2\sin^2\theta \end{cases}_{R_2} = \begin{cases} -L\dot{\psi}^2\sin\theta\cos\theta \\ 2L\dot{\psi}\dot{\theta}\cos\theta \\ -L\dot{\theta}^2 - L\dot{\psi}^2\sin^2\theta \end{cases}_{R_2}$$

Chapter IV

Geometry of Mass

Chapter III: Kinematics of the Rigid Body

III.1 Introduction

A rigid body is an idealization of a body that does not deform or change shape. Formally it is defined as a collection of particles with the property that the distance between particles remains unchanged during the course of motions of the body. Like the approximation of a rigid body as a particle, this is never strictly true. All bodies deform as they move. However, the approximation remains acceptable as long as the deformations are negligible relative to the overall motion of the body.

Kinematics of rigid bodies: relations between time and the positions, velocities, and accelerations of the particles forming a rigid body.

III.2 Fundamental Assumptions

To study the motion of a material point P, or more generally a system of particles or solids, an observer must identify their position:

- In space;
- In time.

In classical kinematics, it is assumed that:

- space is Euclidean (three-dimensional);
- Time is absolute (independent of the observer).

III.3 Reference Frames

To fully study kinematic motion, the observer must define:

- a spatial reference frame linked to the observer with an origin O and an orthonormal basis ($\vec{i}$, $\vec{j}$, $\vec{k}$) forming the trihedron (O, $\vec{i}$, $\vec{j}$, $\vec{k}$), which fully defines the spatial reference frame;
- a time reference (time scale) with an origin and a unit of measurement. In the MKSA system, the second is the unit of time.

The spatial reference frame and the time reference together define the <space-time> reference frame noted as (R). In this frame, at a given moment by the clock, the position of a point r (t) is defined by its coordinates x (t), y (t), z (t) such that:

$$\overrightarrow{Or} = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

The position of point P is known instantaneously in both space and time.

III.4 Motion Relative to Translating Axes

III.4.1 Trajectory

Let point M be identified in a fixed reference frame R (O,$\vec{i}$, $\vec{j}$, $\vec{k}$). Its position is given at each instant t by the vector (Figure III.1):

$$\overrightarrow{r(t)} = \overrightarrow{OM} = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k},$$

The vector $\overrightarrow{r(t)}$ has components in the fixed reference frame at instant t. $\overrightarrow{r(t)} = \begin{cases} x(t) \\ y(t) \\ z(t) \end{cases}$

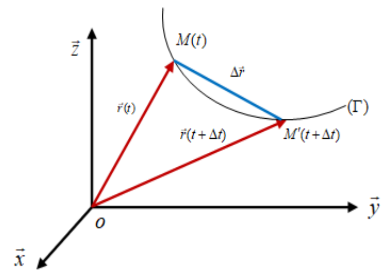


Figure III.1: Trajectory of a point

The displacement of point M in space is given by the parametric equations of coordinates (x, y, z) as functions of time. By eliminating the time parameter, we obtain the trajectory described by this point in space.

$\overrightarrow{r(t)}$= M(t): position of point M in R (O,$\vec{i}$, $\vec{j}$, $\vec{k}$) at instant t.

$\overrightarrow{r(t+\Delta t)}$=M(t+Δt): position of point M in R (O,$\vec{i}$, $\vec{j}$, $\vec{k}$) at instant t+Δt.

The displacement vector from $\overrightarrow{r(t)}$ to $\overrightarrow{r(t+\Delta t)}$ is given by $\Delta\,\overrightarrow{r(t)} = \overrightarrow{r(t+\Delta t)} - \overrightarrow{r(t)}$.

The positions occupied by point M in space describe a trajectory (Γ) with respect to the chosen reference frame R (O,$\vec{i}$, $\vec{j}$, $\vec{k}$).

III.4.2 Velocity Vector

The material point moves from position M(t) to position M(t+Δt) during the time interval Δt at an average speed:

$$\overrightarrow{V_m} = \frac{\overrightarrow{MM'}}{\Delta t} = \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \frac{\Delta\overrightarrow{r(t)}}{\Delta t}$$

The instantaneous velocity vector is obtained when: Δt →0, defined as:

$$\vec{V} = \lim_{\Delta t \to 0} \overrightarrow{V_m} = \lim_{\Delta t \to 0} \frac{\Delta\overrightarrow{r(t)}}{\Delta t} = \frac{d\vec{r}(t)}{\Delta t}$$

This vector is always tangent to the trajectory and directed in the direction of motion.

III.4.3 Acceleration Vector

The derivative of the velocity vector in the same reference frame R (O,$\vec{i}$, $\vec{j}$, $\vec{k}$) gives the instantaneous acceleration of point M:

$$\overrightarrow{\gamma_m} = \frac{\vec{V}(t+\Delta t) - \vec{V}(t)}{\Delta t} = \frac{\Delta \vec{V}(t)}{\Delta t}$$

The instantaneous acceleration is:

$$\vec{\gamma} = \lim_{\Delta t \to 0} \overrightarrow{\gamma_m} = \lim_{\Delta t \to 0} \frac{\Delta \overrightarrow{V(t)}}{\Delta t} = \frac{d\vec{V}(t)}{\Delta t} = \frac{d^2\vec{r}(t)}{dt^2}$$

The two kinematic vectors help to understand the nature of the motion and to predict the different phases, depending on whether the velocity vector is in the same or opposite direction to the acceleration vector.

III.7 Kinematics of the Rigid Body

A perfect rigid body (S) is a set of material elements whose mutual distances do not vary over time. Consequently, the velocities between these points are not independent. Hence, the kinematics of the rigid body deals with the distribution of velocities of points within a body independently of the causes that generated the motion of the solid.

The mechanics of solids allow us to study the behavior of solids and determine all the kinematic parameters of all its points regardless of the nature of the motion. The transport formula allows, by knowing the speed of a single point of the solid, to easily deduce the speed of all points of the solid. The objective of the kinematics of the solid is to know the position, speed, and acceleration of all points of the solid relative to a determined frame of reference.

III.7.1 Concept of Frames and Reference Systems

To study the motion of a solid or a system composed of several solids, it is essential to locate the position of each point as well as the kinematic vectors in space and time. In classical kinematics, we consider that space is three-dimensional Euclidean and time is absolute and independent of the observer. To locate the solid, the observer defines:

- A spatial frame defined by an origin O and an orthonormal basis $(\vec{x}_0, \vec{y}_0, \vec{z}_0)$. The trihedron $(O, \vec{x}_0, \vec{y}_0, \vec{z}_0)$ completely defines the spatial frame in which the coordinates of all points of the solid can be expressed.
- A time frame (also called a time scale) with an origin and a time unit.

In the MKSA system, the unit of time is the second.

These two frames define a space-time frame called a reference frame or simply a frame in classical kinematics. We then choose an arbitrary point O_s on the solid. The position of this point is given at each instant by the position vector $\overrightarrow{OO_s}$ expressed in the frame R $(O, \vec{x}_0, \vec{y}_0, \vec{z}_0)$. The coordinates of the point O_s depend on time and allow us to know at any moment the position of the frame R $(O, \vec{x}_s, \vec{y}_s, \vec{z}_s)$ linked to the solid. The transition from the frame R $(O, \vec{x}_0, \vec{y}_0, \vec{z}_0)$ to the frame R $(O, \vec{x}_s, \vec{y}_s, \vec{z}_s)$ linked to the solid is determined by the transition matrix, which expresses the unit vectors $(O, \vec{x}_0, \vec{y}_0, \vec{z}_0)$ in terms of the unit vectors $(O, \vec{x}_s, \vec{y}_s, \vec{z}_s)$. This transition matrix is expressed in terms of Euler angles. The orientation of the frame linked to the solid is independent of the choice of the point O_s.

The set of translation and rotation parameters constitute the situation parameters or degrees of freedom of the solid in space relative to the frame $R(O, \vec{x}_0, \vec{y}_0, \vec{z}_0)$. If the number of parameters is equal to 6 (3 rotations and 3 translations), the solid is said to be completely free in $R(O, \vec{x}_0, \vec{y}_0, \vec{z}_0)$. If the number of parameters is less than 6, the solid is said to be constrained or subjected to constraints where certain parameters do not vary over time.

III.7.2 Notation Systems

In the study of kinematics, we adopt the following notation:

Let $R_i(O, \vec{x}_i, \vec{y}_i, \vec{z}_i)$ be a frame linked to the observer and P a point of the solid:

- $\overrightarrow{O_iP}$: Position vector of point P relative to frame R_i;
- $\vec{V}^i(P) = \dfrac{d^i\overrightarrow{O_iP}}{dt}$: Speed of point P relative to frame R_i;
- $\vec{\gamma}^i(P) = \dfrac{d^i\vec{V}^i(P)}{dt}$: Acceleration of point P relative to frame R_i.

The kinematic parameters are always linked to the frame. The kinematic parameters (velocity and acceleration vectors) of the points of the solid are studied in a frame $R_i(O, \vec{x}_i, \vec{y}_i, \vec{z}_i)$ linked to the observer. This frame is called the study frame.

The components of the velocity $\vec{V}^i(P)$ and acceleration vectors $\vec{\gamma}^i(P)$ being measured and defined in the frame $R_i(O, \vec{x}_i, \vec{y}_i, \vec{z}_i)$, we can know their components in any frame of space $R_p(O, \vec{x}_P, \vec{y}_P, \vec{z}_P)$, which we will call the projection frame.

Choosing this projection frame allows us to express the kinematic parameters with simpler mathematical expressions. It is often interesting to choose the projection frame different from the study frame to simplify and reduce calculations. The projection frame being mobile relative to the study frame, care must be taken during derivations as the unit vectors of the projection frame change direction and this must be accounted for.

III.7.3 Motion of a Frame R_k Relative to a Frame R_i Linked to the Observer

Let $R_i(O_i, \vec{x}_i, \vec{y}_i, \vec{z}_i)$ be a frame linked to the observer and $R_k(O_k, \vec{x}_k, \vec{y}_k, \vec{z}_k)$ a frame in any motion relative to the first. Any point in space can be completely located in R_k and its components deduced in R_i or conversely by knowing the motion of R_k relative to R_i. The motion of the frame R_k is completely known if:

- The position of its center O_k is completely known in R_i;
- The orientation of the axes of R_k is known relative to those of R_i.

III.7.3.1 Location of the Center O_k of the Frame R_k

The location of the center point O_k of the frame R_k is determined by the components of the vector $\overrightarrow{O_iO_k}$ linking the two centers of the frames in R_i or R_k, which results in the following relations:

In R_i: $\begin{cases} \overrightarrow{O_iO_k}.\vec{x}_i \\ \overrightarrow{O_iO_k}.\vec{y}_i \\ \overrightarrow{O_iO_k}.\vec{z}_i \end{cases}_{R_i}$ In R_k: $\begin{cases} \overrightarrow{O_iO_k}.\vec{x}_k \\ \overrightarrow{O_iO_k}.\vec{y}_k \\ \overrightarrow{O_iO_k}.\vec{z}_k \end{cases}_{R_k}$

III.7.3.2 Formula for the Mobile Basis

Let $R_i(O_i, \vec{x}_i, \vec{y}_i, \vec{z}_i)$ be a fixed frame and $R_k(O_k, \vec{x}_k, \vec{y}_k, \vec{z}_k)$ a frame mobile relative to the first. The unit vectors of the frame R_k are orthogonal to each other and have constant modules equal to 1, but they change direction in space.

$\|\vec{x}_k\| = \|\vec{y}_k\| = \|\vec{z}_k\| = 1$ and $\vec{x}_k.\vec{y}_k = 0$, $\vec{x}_k.\vec{z}_k = 0$, $\vec{y}_k.\vec{z}_k = 0$

So we have: $\dfrac{d^i\vec{x}_k}{dt} = \vec{\Omega}_k^i \wedge \vec{x}_k$; $\dfrac{d^i\vec{y}_k}{dt} = \vec{\Omega}_k^i \wedge \vec{y}_k$; $\dfrac{d^i\vec{z}_k}{dt} = \vec{\Omega}_k^i \wedge \vec{z}_k$

III.7.3.3 Derivative in the Frame R_i of a Vector Expressed in a Frame R_k

The vector $\vec{V}(t)$ can be written as $\vec{V}(t) = X_k\vec{x}_k + Y_k\vec{y}_k + Z_k\vec{z}_k$ in the frame R_k.

Its derivative in the frame R_k is expressed as: $\dfrac{d^k\vec{V}(t)}{dt} = \dot{X}_k\vec{x}_k + \dot{Y}_k\vec{y}_k + \dot{Z}_k\vec{z}_k$

Its derivative in the frame R_i is written as:

$$\frac{d^i\vec{V}(t)}{dt} = \frac{d^k\vec{V}(t)}{dt} + X_k\vec{\Omega}_k^i\vec{x}_k + Y_k\vec{\Omega}_k^i\vec{y}_k + Z_k\vec{\Omega}_k^i\vec{z}_k$$

$$\frac{d^i\vec{V}(t)}{dt} = \frac{d^k\vec{V}(t)}{dt} + \vec{\Omega}_k^i \wedge (X_k\vec{x}_k + Y_k\vec{y}_k + Z_k\vec{z}_k = \frac{d^k\vec{V}(t)}{dt} + \vec{\Omega}_k^i \wedge \vec{V}(t)$$

Finally, we obtain: $\dfrac{d^i\vec{V}(t)}{dt} = \dfrac{d^k\vec{V}(t)}{dt} + \vec{\Omega}_k^i \wedge \vec{V}(t)$

III.7.3.4 Properties of the Vector $\vec{\Omega}_k^i$

a) The vector $\vec{\Omega}_k^i$ is antisymmetric with respect to indices i and j: $\vec{\Omega}_k^i = -\vec{\Omega}_i^k$

b) Chasles' formula: $\vec{\Omega}_k^i = \vec{\Omega}_k^j + \vec{\Omega}_j^i$ (principle of composition)

c) $\dfrac{d^i\vec{\Omega}_k^i}{dt} = \dfrac{d^k\vec{\Omega}_k^i}{dt}$ Equality of derivatives with respect to indices.

III.7.4 Transition Matrix (Type 1 Euler Angles)

Let $R_i(O_i,\vec{x}_i,\vec{y}_i,\vec{z}_i)$ be a fixed frame and $R_k(O_k,\vec{x}_k,\vec{y}_k,\vec{z}_k)$ a frame linked to the solid (S) in any motion in space. The center O_k of the frame R_k belongs to the solid $O_k \in$ (S). In the case of type 1 Euler angles, we consider that the centers O_i and O_k of the two frames are coincident: $O_i \equiv O_k$, which means that the frame R_k only undergoes rotations relative to the frame R_i. Three independent parameters are necessary to completely define the orientation of the frame R_k relative to that of R_i.

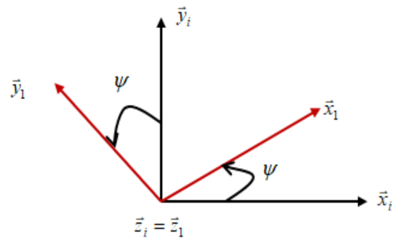
The transition from frame R_k to frame R_i is achieved by three rotations using two intermediate frames R_1 and R_2.

III.7.4.1 Transition from Frame R₁ to Frame Rᵢ: (the yaw rotation)

The rotation is performed around the axis $\vec{z}_i \equiv \vec{z}_1$.

We transition from frame $R_i(O_i,\vec{x}_i,\vec{y}_i,\vec{z}_i)$ to frame $R_1(O_1,\vec{x}_1,\vec{y}_1,\vec{z}_1)$ by rotating by an angle ψ: called the precession angle. The rotation speed is given by:

$$\vec{\Omega}_1^i = \dot{\psi}\vec{z}_i = \dot{\psi}\vec{z}_1 \text{ Because } \vec{z}_i \text{ is confused with } \vec{z}_1$$



$\psi = (\vec{x}_i,\vec{x}_1) = (\vec{y}_i,\vec{y}_1)$ Avec $\vec{z}_1 = \vec{x}_1 \wedge \vec{y}_1$

The representation is done by plane figures from which we construct the transition matrices. Thus, we have:

$$\vec{x}_1 = \cos\psi\vec{x}_i + \sin\psi\vec{y}_i + 0.\vec{z}_i$$
$$\vec{y}_1 = -\sin\psi\vec{x}_i + \cos\psi\vec{y}_i + 0.\vec{z}_i$$
$$\vec{z}_1 = 0.\vec{x}_i + 0.\vec{y}_i + \vec{z}_i$$

These three equations can be written in matrix form, and we obtain:

$$\begin{pmatrix}\vec{x}_1\\ \vec{y}_1\\ \vec{z}_1\end{pmatrix} = \begin{pmatrix}\cos\psi & \sin\psi & 0\\ -\sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{pmatrix}.\begin{pmatrix}\vec{x}_i\\ \vec{y}_i\\ \vec{z}_i\end{pmatrix}$$

$$P_{R_1\to R_i} = \begin{pmatrix}\cos\psi & \sin\psi & 0\\ -\sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{pmatrix}$$ This is the transition matrix from frame R_1 to frame R_i.

The transition matrix from R_i to R_1 is equal to the transpose of the above matrix $P_{R_1\to R_i}$: $P_{R_i\to R_1} = P^T{}_{R_1\to R_i}$.

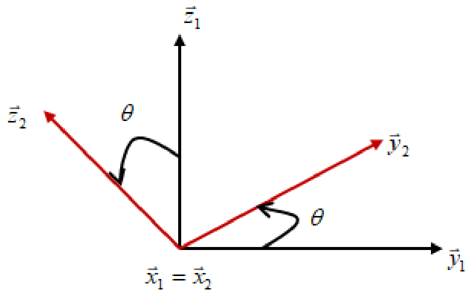
III.7.4.2 Transition from Frame R₂ to Frame R₁: (the pitch rotation)

The rotation is performed around the axis $\vec{x}_1 \equiv \vec{x}_2$.

We transition from frame $R_2(O_2,\vec{x}_2,\vec{y}_2,\vec{z}_2)$ to frame $R_1(O_1,\vec{x}_1,\vec{y}_1,\vec{z}_1)$ by rotating by an angle θ: called the nutation angle. The rotation speed is given by:

$\vec{\Omega}_2^1 = \dot{\theta}\vec{x}_1 = \dot{\theta}\vec{x}_2$

$\vec{\Omega}_2^1 = \dot{\theta}\vec{x}_1 = \dot{\theta}\vec{x}_2$ Because $\vec{x}_1$ is confused with $\vec{x}_2$



$\theta = (\vec{y}_1, \vec{y}_2) = (\vec{z}_1, \vec{z}_2)$ Avec $\vec{x}_2 = \vec{y}_2 \wedge \vec{z}_2$

Thus, we have:

$$\vec{x}_2 = \vec{x}_i + 0.\vec{y}_1 + 0.\vec{z}_1$$
$$\vec{y}_2 = 0.\vec{x}_1 + \cos\theta\vec{y}_1 + \sin\theta\vec{z}_i$$
$$\vec{z}_1 = 0.\vec{x}_1 - \sin\theta\vec{y}_1 + \cos\theta\vec{z}_i$$

In matrix form we get:

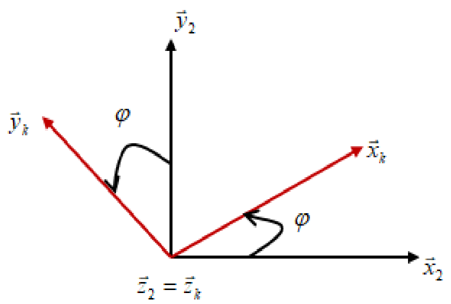
$$\begin{pmatrix} \vec{x}_1 \\ \vec{y}_1 \\ \vec{z}_1 \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} \vec{x}_i \\ \vec{y}_i \\ \vec{z}_i \end{pmatrix}$$

$$P_{R_1 \to R_i} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ This is the transition matrix from frame R_2 to frame R_1

III.7.4.3 Transition from Frame R_k to Frame R_2: (the roll rotation)

The rotation is performed around the axis $\vec{z}_2 \equiv \vec{z}_k$.

We transition from frame R_k to frame R_2 by rotating by an angle φ: called the proper rotation angle. The rotation speed is given by:



$\varphi = (\vec{x}_2, \vec{x}_k) = (\vec{y}_2, \vec{y}_k)$ Avec $\vec{z}_k = \vec{x}_k \wedge \vec{y}_k$

$\vec{\Omega}_k^2 = \dot{\varphi}\vec{z}_2 = \dot{\varphi}\vec{z}_k$ Because $\vec{z}_i$ is confused with $\vec{z}_1$

Thus, we have:

$$\vec{x}_k = \cos\varphi\vec{x}_2 + \sin\varphi\vec{y}_2 + 0.\vec{z}_2$$
$$\vec{y}_k = -\sin\varphi\vec{x}_2 + \cos\varphi\vec{y}_2 + 0.\vec{z}_2$$
$$\vec{z}_k = 0.\vec{x}_2 + 0.\vec{y}_2 + \vec{z}_2$$

In matrix form we get:

$$\begin{pmatrix} \vec{x}_k \\ \vec{y}_k \\ \vec{z}_k \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} \vec{x}_2 \\ \vec{y}_2 \\ \vec{z}_2 \end{pmatrix}$$

$$P_{R_k \to R_2} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ This is the transition matrix from frame R_k to frame R_2

The passage from the R_k reference to the R_i reference or vice versa is done by three successive rotations such that all the axes of R_k occupy positions different from that of R_i. The transition matrix from R_k to R_i is given by the product of the three successive matrices, we obtain:

$$\begin{pmatrix} \vec{x}_k \\ \vec{y}_k \\ \vec{z}_k \end{pmatrix} = \begin{pmatrix} \cos\varphi\cos\psi - \sin\varphi\cos\theta\sin\psi & \cos\varphi\sin\psi + \sin\varphi\cos\theta\cos\psi & \sin\varphi\sin\theta \\ -\sin\varphi\cos\psi - \sin\psi\cos\theta\cos\varphi & -\sin\varphi\sin\psi + \cos\varphi\cos\theta\cos\psi & \cos\varphi\sin\theta \\ \sin\theta\sin\psi & -\sin\theta\cos\psi & \cos\theta \end{pmatrix} . \begin{pmatrix} \vec{x}_i \\ \vec{y}_i \\ \vec{z}_i \end{pmatrix}$$

The transition matrix from R_i to R_k is given by the transpose of the latter.

The instantaneous rotation vector of the reference frame R_k with respect to R_i will have the vector expression:

$$\vec{\Omega}_k^i = \dot{\psi}\vec{z}_i + \dot{\theta}\vec{x}_1 + \dot{\varphi}\vec{z}_2$$

It will have a different expression depending on whether it is written in one or the other of the two markers.

In R_i, we will have: $\vec{\Omega}_k^i = \left\{ \begin{array}{c} \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ -\dot{\varphi}\sin\theta\cos\psi + \dot{\theta}\sin\psi \\ \dot{\varphi}\cos\psi + \dot{\psi} \end{array} \right.{}_{R_i}$

In R_k, we will have: $\vec{\Omega}_k^i = \left\{ \begin{array}{c} \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi \\ \dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi \\ \dot{\varphi} + \dot{\psi}\cos\psi \end{array} \right.{}_{R_k}$

This instantaneous rotation vector allows deducing the speed of all the solid points by knowing the speed of a single point belonging to the solid.

III.8 Fields of Velocity and Acceleration of a Solid

Consider a fixed reference frame R_i $(O_i, \vec{x}_i, \vec{y}_i, \vec{z}_i)$ and a solid (S_k) linked to a moving reference frame R_k $(O_k, \vec{x}_k, \vec{y}_k, \vec{z}_k)$ in space. For any point on the solid (S_k), we can associate its position vector, thus its velocity vector and acceleration vector.

Consider two points A_k and B_k belonging to the solid (S_k). We will seek a relationship between their velocities and their accelerations.

III.8.1 Velocity Fields

The solid (S_k) is non-deformable, so the distance $\overrightarrow{A_k B_k} = Cte$ remains constant over time in both reference frames. This vector will be expressed differently in R_i and R_k. The velocities of points A_K and B_k are different because the solid has arbitrary motion.

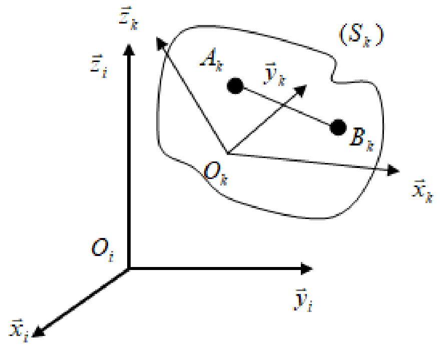


Figure III. 7: Velocity fields

In reference frame R_i: $\overrightarrow{O_i B_k} = \overrightarrow{O_i A_k} + \overrightarrow{A_k B_k} \Rightarrow \overrightarrow{A_k B_k} = \overrightarrow{O_i B_k} - \overrightarrow{O_i A_k} = Cte$

In reference frame R_k: $\overrightarrow{O_k B_k} = \overrightarrow{O_k A_k} + \overrightarrow{A_k B_k} \Rightarrow \overrightarrow{A_k B_k} = \overrightarrow{O_k B_k} - \overrightarrow{O_k A_k} = Cte$

From these two expressions, we can deduce a relationship between the velocities of the two points belonging to the solid.

The velocities of the two points with respect to the reference frame R_i are given by:

$\vec{V}^i(A_k) = \dfrac{d^i \overrightarrow{O_i A_K}}{dt}$ and $\vec{V}^i(B_k) = \dfrac{d^i \overrightarrow{O_i B_K}}{dt}$

These two expressions can be written as:

$$\vec{V}^i(A_k) = \frac{d^i \overrightarrow{O_i A_K}}{dt} = \frac{d^k \overrightarrow{O_i A_K}}{dt} + \vec{\Omega}_k^i \wedge \overrightarrow{O_i A_k} \quad \ldots\ldots\ldots\ldots(1)$$

$$\vec{V}^i(B_k) = \frac{d^i \overrightarrow{O_i B_K}}{dt} = \frac{d^k \overrightarrow{O_i B_K}}{dt} + \vec{\Omega}_k^i \wedge \overrightarrow{O_i B_k} \quad \ldots\ldots\ldots\ldots(2)$$

By subtracting the two expressions (2) - (1):

$$\vec{V}^i(B_k) - \vec{V}^i(A_k) = \frac{d^i (\overrightarrow{O_i B_K} - \overrightarrow{O_i A_K})}{dt} + \vec{\Omega}_k^i \wedge (\overrightarrow{O_i B_k} - \overrightarrow{O_i A_k})$$

We know that: $\dfrac{d^i (\overrightarrow{O_i B_K} - \overrightarrow{O_i A_K})}{dt} = \dfrac{d^i \overrightarrow{A_k B_k}}{dt} = 0$ because $\overrightarrow{O_i B_k} - \overrightarrow{O_i A_k} = \overrightarrow{A_k B_k}$

Thus, we obtain the distribution relationship of velocities in a solid: $\vec{V}^i(B_k) = \vec{V}^i(A_k) + \vec{\Omega}_k^i \wedge (\overrightarrow{A_k B_k})$

This relationship is of great importance in the kinematics and dynamics of solids. It allows us, from the velocity of one point of the solid, to deduce the velocity of all other points of the solid by knowing the rotational velocity of the associated reference frame.

Note:

a) If the rotation vector is zero $\Omega_k^i = 0$, then the solid is in pure translation motion, and all points of the solid have the same velocity: $\vec{V}^i(B_k) = \vec{V}^i(A_k)$;

b) If $\vec{V}^i(A_k) = 0$ and $\vec{V}^i(B_k) = \vec{\Omega}_k^i \wedge (\overrightarrow{A_k B_k})$, the solid is in pure rotational motion around the point $A_k \in (S_k)$;

c) The general motion of a solid can be described as a composition of a translation motion of point $A_k \in (S_k)$ at velocity $\vec{V}^i(A_k)$ and a rotational motion around point $A_k \in (S_k)$ at rotational velocity Ω_k^i.

III.8.2 Equiprojectivity of the Velocity Field of a Solid

We can demonstrate it in two different methods.

a) Previously, we showed that: $\vec{V}^i(B_k) = \vec{V}^i(A_k) + \vec{\Omega}_k^i \wedge (\overrightarrow{A_k B_k})$

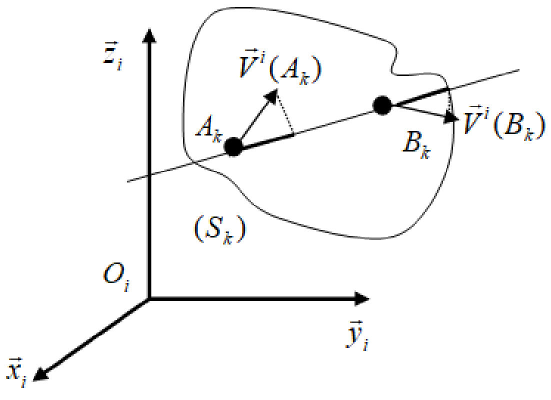


Figure III.8 : Equiprojectivity of the velocity field

By multiplying this expression by the vector $\overrightarrow{A_k B_k}$, we obtain:

$$\overrightarrow{A_k B_k}.\vec{V}^i(B_k) = \overrightarrow{A_k B_k}.\vec{V}^i(A_k) + \overrightarrow{A_k B_k}.(\vec{\Omega}_k^i \wedge \overrightarrow{A_k B_k})$$

By circular permutation of the mixed product, we can easily see that the expression:

$$\overrightarrow{A_k B_k}.(\vec{\Omega}_k^i \wedge \overrightarrow{A_k B_k}) = \vec{\Omega}_k^i.(\overrightarrow{A_k B_k} \wedge \overrightarrow{A_k B_k}) = \vec{0}$$

Thus, we obtain the equality: $\overrightarrow{A_k B_k}.\vec{V}^i(B_k) = \overrightarrow{A_k B_k}.\vec{V}^i(A_k)$

(Property of Equiprojectivity of the Velocity Field of the Solid)

b) This expression can be found another way. The solid is non-deformable, and the distance $\overrightarrow{A_k B_k}$ is constant, thus:

$$\frac{d(\overrightarrow{A_k B_k})^2}{dt} = 0$$

$$\frac{d(\overrightarrow{A_k B_k})^2}{dt} = 2\overrightarrow{A_k B_k}\frac{d\overrightarrow{A_k B_k}}{dt} = 0$$

$2\overrightarrow{A_k B_k}(\vec{V}^i(B_k) - \vec{V}^i(A_k)) = 0$ From where: $\overrightarrow{A_k B_k}.\vec{V}^i(B_k) = \overrightarrow{A_k B_k}.\vec{V}^i(A_k)$

This equiprojectivity property implies the existence of a free vector $\vec{\Omega}_k^i$ such that: $\vec{V}^i(B_k) = \vec{V}^i(A_k) + \vec{\Omega}_k^i \wedge (\overrightarrow{A_k B_k})$ which allows us to introduce the notion of kinematic screw.

III.8.3 Acceleration Fields

For each point of the solid (S_k) linked to the reference frame R_k, we deduce the acceleration from the velocity using the relation: $\vec{\gamma}^i(A_k) = \dfrac{d^i\vec{V}^i(A_k)}{dt}$

We will find a relationship linking the accelerations: $\vec{\gamma}^i(A_k)$ and $\vec{\gamma}^i(B_k)$.

We have already established a relationship between the velocities of the two points: $\vec{V}^i(B_k) = \vec{V}^i(A_k) + \vec{\Omega}_k^i \wedge (\overrightarrow{A_k B_k})$

We deduce the relationship between the accelerations by differentiating the expression of velocities.

$$\vec{\gamma}^i(B_k) = \frac{d^i\vec{V}^i(B_k)}{dt} = \frac{d^i\vec{V}^i(A_k)}{dt} + \frac{d^i\vec{\Omega}_k^i}{dt} \wedge \overrightarrow{A_k B_k} + \vec{\Omega}_k^i \wedge \frac{d^i\overrightarrow{A_k B_k}}{dt}$$

And since: $\dfrac{d^i\overrightarrow{A_k B_k}}{dt} = \dfrac{d^k\overrightarrow{A_k B_k}}{dt} + \vec{\Omega}_k^i \wedge \overrightarrow{A_k B_k} = \vec{\Omega}_k^i \wedge \overrightarrow{A_k B_k}$ because $\dfrac{d^k\overrightarrow{A_k B_k}}{dt} = \vec{0}$

Finally, we obtain the relation between the accelerations of the two points A_k and B_k of the solid: $\vec{\gamma}^i(B_k) = \vec{\gamma}^i(A_k) + \dfrac{d^i\vec{\Omega}_k^i}{dt} \wedge \overrightarrow{A_k B_k} + \vec{\Omega}_k^i \wedge (\vec{\Omega}_k^i \wedge \overrightarrow{A_k B_k})$

We observe that if the rotational velocity is constant $\vec{\Omega}_k^i = \vec{0}$, the expression becomes:

$$\vec{\gamma}^i(B_k) = \vec{\gamma}^i(A_k) + \vec{\Omega}_k^i \wedge (\vec{\Omega}_k^i \wedge \overrightarrow{A_k B_k}) = \vec{\gamma}^i(A_k) - \overrightarrow{A_k B_k}(\vec{\Omega}_k^i)^2$$

III.8.4 Kinematic Screw

The distribution formula of velocities is given by the relation: $\vec{V}^i(B_k) = \vec{V}^i(A_k) + \vec{\Omega}_k^i \wedge (\overrightarrow{A_k B_k})$

The transport formula of moments between two points A_k and B_k of the solid is expressed as: $\vec{M}(B_k) = \vec{M}(A_k) + \vec{R} \wedge \overrightarrow{A_k B_k}$

We note that there is equivalence between these two equations. The velocity vector at point B_k is the moment at point B_k of a screw, which we will denote as $[C]_{B_k}$, and the resultant is none other than the instantaneous rotation vector $\vec{\Omega}_k^i$.

The kinematic screw at point B_k (or the distribution screw of velocities) relative to the motion of the solid with respect to R_i has the reduced elements:

- Instantaneous rotation vector: $\vec{\Omega}_k^i$
- Velocity at point B_k: $\vec{V}^i(B_k)$

It will be noted in the form : $[C]_{B_k} = \begin{cases} \vec{\Omega}_k^i \\ \vec{V}^i(B_k) = \vec{V}^i(A_k) + \vec{\Omega}_k^i \wedge \overrightarrow{A_k B_k} \end{cases}$

The kinematic torso is of great interest because it completely characterizes the motion of a solid relative to the Ri mark with regard to speeds. As the reduction elements of the kinematic torso are time functions, and then the kinematic torso depends on it, so it has at every moment a different result and velocity field.

III.9 Laws of Composition of motion

III.9.1 Law of Composition of Velocities

Consider $R_i(O_i, \vec{x}_i, \vec{y}_i, \vec{z}_i)$ a fixed reference frame and $R_k(O_k, \vec{x}_k, \vec{y}_k, \vec{z}_k)$ a reference frame moving arbitrarily with respect to the fixed frame. We consider a solid (S_k) whose motion is known in the relative frame $R_k(O_k, \vec{x}_k, \vec{y}_k, \vec{z}_k)$.

Let P be a point on the solid. We can write at any moment:

$$\overrightarrow{O_i P} = \overrightarrow{O_i O_k} + \overrightarrow{O_k P}$$

The velocity of point P in the frame R_i is given by the derivative of the vector $\overrightarrow{O_i P}$ in the same frame. $\vec{V}^i(P) = \frac{d^i \overrightarrow{O_i P}}{dt} = \frac{d^i \overrightarrow{O_i O_k}}{dt} + \frac{d^i \overrightarrow{O_k P}}{dt}$

Developing the two terms of velocity gives:

$\frac{d^i \overrightarrow{O_i O_k}}{dt} = \vec{V}^i(O_k)$: velocity of the center of the frame R_k with respect to the frame R_i.

$$\frac{d^i \overrightarrow{O_k P}}{dt} = \frac{d^k \overrightarrow{O_k P}}{dt} + \vec{\Omega}_k^i \wedge \overrightarrow{O_k P} = \vec{V}^k(P) + \vec{\Omega}_k^i \wedge \overrightarrow{O_k P}$$

Finally, the velocity of point P in the frame R_i is written as: $\vec{V}^i(P) = \vec{V}^k(P) + (\vec{V}^i(O_k) + \vec{\Omega}_k^i \wedge \overrightarrow{O_k P})$

This can also be written in the form: $\vec{V}^i(P) = \vec{V}^k(P) + \vec{V}^i(O_k) + \vec{V}_k^i(P)$

where:

$\vec{V}^i(P)$: Absolute velocity of point P for an observer in R_i

$\vec{V}^k(P)$: Relative velocity of point P with respect to R_k moving with respect to R_i

$\vec{V}_k^i(P)$: transport velocity of point P if it were stationary in R_k.

Note:

$\vec{V}_k^i(P) = -\vec{V}_i^k(P)$: Antisymmetric with respect to the indices, and thus to the reference frames.

$\vec{V}_k^i(P) = \vec{V}_k^j(P) + \vec{V}_j^i(P)$

III.9.2 Law of Composition of Accelerations

The absolute acceleration $\gamma^i(P)$ of point P is derived from the absolute velocity:

$$\vec{\gamma}^i(P) = \frac{d^i \vec{V}^i(P)}{dt} = \frac{d^i \vec{V}^k(P)}{dt} + \frac{d^i \vec{V}^i(O_k)}{dt} + \frac{d^i(\vec{\Omega}_k^i \wedge \overrightarrow{O_k P})}{dt}$$

Developing each of the three terms:

1. $\dfrac{d^i \vec{V}^i(P)}{dt} = \dfrac{d^k \vec{V}^k(P)}{dt} + \vec{\Omega}_k^i \wedge \vec{V}^k(P) = \vec{\gamma}^k(P) + \vec{\Omega}_k^i \wedge \vec{V}^k(P)$;

2. $\dfrac{d^i \vec{V}^i(O_k)}{dt} = \vec{\gamma}^i(O_k)$;

3. $\dfrac{d^i(\vec{\Omega}_k^i \wedge \overrightarrow{O_k P})}{dt} = \dfrac{d^i \vec{\Omega}_k^i}{dt} \wedge \overrightarrow{O_k P} + \vec{\Omega}_k^i \wedge \dfrac{d^i \overrightarrow{O_k P}}{dt}$

$$\frac{d^i(\vec{\Omega}_k^i \wedge \overrightarrow{O_kP})}{dt} = \frac{d^i\vec{\Omega}_k^i}{dt} \wedge \overrightarrow{O_kP} + \vec{\Omega}_k^i \wedge \frac{d^i\overrightarrow{O_kP}}{dt} = \frac{d^i\vec{\Omega}_k^i}{dt} \wedge \overrightarrow{O_kP} + \vec{\Omega}_k^i \wedge (\frac{d^k\overrightarrow{O_kP}}{dt} + \vec{\Omega}_k^i \wedge \overrightarrow{O_kP})$$
$$= \frac{d^i\vec{\Omega}_k^i}{dt} \wedge \overrightarrow{O_kP} + \vec{\Omega}_k^i \wedge (\vec{V}^k(P) + \vec{\Omega}_k^i \wedge \overrightarrow{O_kP})$$

Summing the three terms gives:

$$\vec{\gamma}^i(P) = \vec{\gamma}^k(P) + \vec{\Omega}_k^i \wedge \vec{V}^k(P) + \vec{\gamma}^i(O_k) = \frac{d^i\vec{\Omega}_k^i}{dt} \wedge \overrightarrow{O_kP} + \vec{\Omega}_k^i \wedge (\vec{V}^k(P) + \vec{\Omega}_k^i \wedge \overrightarrow{O_kP})$$

$$\vec{\gamma}^i(P) = \vec{\gamma}^k(P) + \left(\vec{\gamma}^i(O_k) + \frac{d^i\vec{\Omega}_k^i}{dt} \wedge \overrightarrow{O_kP} + \vec{\Omega}_k^i \wedge (\vec{\Omega}_k^i \wedge \overrightarrow{O_kP})\right) + 2\vec{\Omega}_k^i \wedge \vec{V}^k(P)$$

This expression can be written in a reduced form:

$$\vec{\gamma}^i(P) = \vec{\gamma}^k(P) + \vec{\gamma}_k^i(P) + \vec{\gamma}_c(P)$$

where:

$\vec{\gamma}^i(P)$: Absolute acceleration of point P (with respect to fixed R_i)

$\vec{\gamma}^k(P)$: Relative acceleration of point P (with respect to frame R_k)

$\vec{\gamma}_k^i(P) = \vec{\gamma}^i(O_k) + \frac{d^i\vec{\Omega}_k^i}{dt} \wedge \overrightarrow{O_kP} + \vec{\Omega}_k^i \wedge (\vec{\Omega}_k^i \wedge \overrightarrow{O_kP})$: Transport acceleration of frame R_k

$\vec{\gamma}_c(P) = 2\vec{\Omega}_k^i \wedge \vec{V}^k(P)$: Coriolis acceleration (complementary acceleration).

The Coriolis acceleration is a composition between the rotational velocity $\vec{\Omega}_k^i$ of the frame R_k with respect to the frame R_i and the relative velocity $\vec{V}^k(P)$ of point P.

The Coriolis acceleration of point P is zero if and only if:

- The rotational velocity of the relative frame with respect to the absolute frame is zero: $\vec{\Omega}_k^i = \vec{0}$;
- The relative velocity of point P is zero: $\vec{V}^k(P) = \vec{0}$;
- The rotational velocity is collinear with the relative velocity: $\vec{\Omega}_k^i // \vec{V}^k(P)$.

Application Exercises

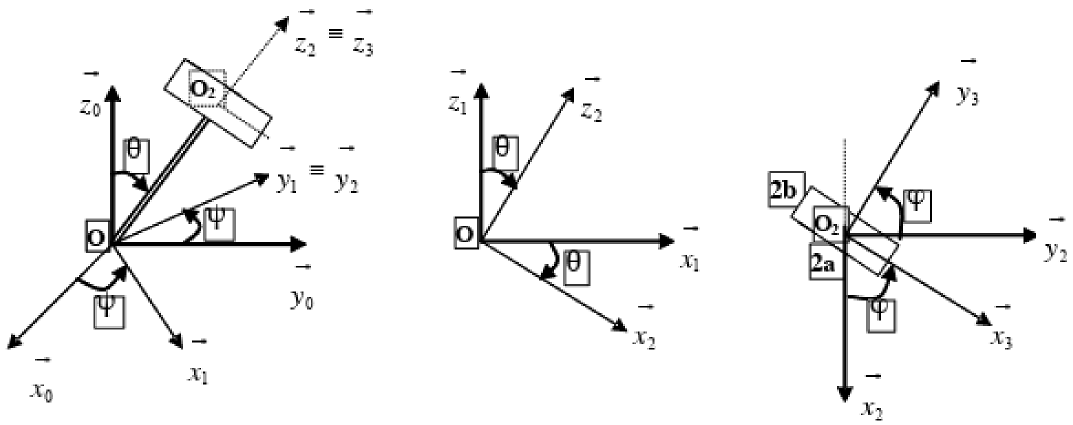
Exercise 01:

Consider the mechanical system composed of a rod O_2 of length L and a rectangular plate of dimensions 2a and 2b hinged at O_2 with the rod (see figure). R_0 being the fixed frame; R_1 rotating by Ψ around the axis $\vec{z}_0$. The plate rotates around the rod at an angular velocity $\dot{\varphi}$.

Given: $\dot{\psi}$ = Cte ; $\dot{\theta}$ = Cte ; $\dot{\varphi}$ = Cte

Determine:

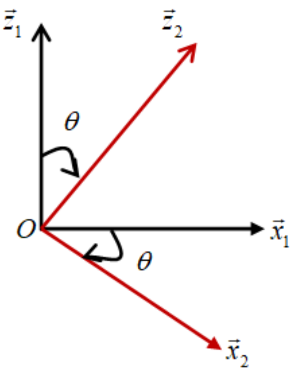
1. The transformation matrices from R_1 to R_2 and from R_3 to R_2;
2. The instantaneous rotation vector of R_3 relative to R_0 expressed in R_2;
3. The velocity $\vec{V}^0(O_2)$ expressed in frame R_2 by differentiation;
4. The velocity $\vec{V}^0(A)$ with respect to R_0 expressed in R_2 by the solid's kinematics;
5. The acceleration expressed $\vec{\gamma}^0(O_2)$ in frame R_2 by differentiation and by the solid's kinematics.



Solution 01:

The rod: OO_2=L; The plate: Length 2a, Width 2b

$R_0(O,\vec{x}_0,\vec{y}_0,\vec{z}_0)$: Fixed frame;



$R_1(O,\vec{x}_1,\vec{y}_1,\vec{z}_1)$: Frame rotating around the axis $\vec{z}_0$ relative to R_0;

$R_2(O,\vec{x}_2,\vec{y}_2,\vec{z}_2)$: Frame attached to the rod rotating around the axis $\vec{y}_1$ relative to R_1;

$R_3(O,\vec{x}_3,\vec{y}_3,\vec{z}_3)$: Frame attached to the plate rotating around the axis $\vec{z}_2$ relative to R_2;

Given: $\dot{\psi}$ = Cte ; $\dot{\theta}$ = Cte ; $\dot{\varphi}$ = Cte

1.Transformation Matrices

Transformation matrix from R_2 to R_1:

$$\begin{pmatrix} \vec{x}_1 \\ \vec{x}_2 \\ \vec{x}_3 \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \vec{x}_2 \\ \vec{y}_2 \\ \vec{z}_2 \end{pmatrix}$$

$P_{R_1 \to R_2}$

Transformation matrix from R_3 to R_2:

$$\begin{pmatrix} \vec{x}_3 \\ \vec{x}_3 \\ \vec{x}_3 \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\kappa & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{x}_2 \\ \vec{y}_2 \\ \vec{z}_2 \end{pmatrix}$$

$P_{R_3 \to R_2}$

2. Instantaneous Rotation Vector of R_3 Relative to R_0 Expressed in R_2

According to Chasles' theorem, we can write:

$$\vec{\Omega}_3^0 = \vec{\Omega}_3^2 + \vec{\Omega}_2^1 + \vec{\Omega}_1^0 = \dot{\varphi}.\vec{z}_2 + \dot{\theta}.\vec{y}_2 + \dot{\psi}.\vec{z}_1$$

Expressing the unit vector $\vec{z}_1$ in frame R_2, we get: $\vec{z}_1 = -\sin\theta\,\vec{x}_2 + \cos\theta\,\vec{z}_2$

$$\vec{\Omega}_3^0 = \dot{\varphi}.\vec{z}_2 + \dot{\theta}.\vec{y}_2 + \dot{\psi}(-\sin\theta\,\vec{x}_2 + \cos\theta\,\vec{z}_2) = -\dot{\psi}\sin\theta\,\vec{x}_2 + \dot{\theta}.\vec{y}_2 + (\dot{\varphi} + \dot{\psi}\cos\theta)\vec{z}_2$$

$$\vec{\Omega}_3^0 = \begin{cases} -\dot{\psi}\sin\theta \\ \dot{\theta} \\ \dot{\varphi} + \dot{\psi}\cos\theta \end{cases}_{R_2}$$

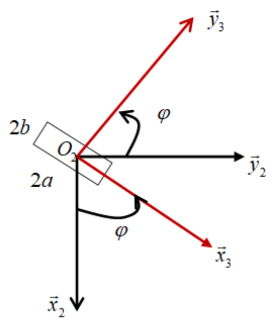
3. $\vec{V}^0(O_2)$ Velocity Expressed in Frame R_2 by Differentiation

By differentiation: $\vec{V}^0(O_2) = \dfrac{d^0\overrightarrow{OO_2}}{dt} = \dfrac{d^2\overrightarrow{OO_2}}{dt} + \vec{\Omega}_2^0 \wedge \overrightarrow{OO_2}$

$$\overrightarrow{OO_2} = \begin{cases} 0 \\ 0 \\ L \end{cases}_{R_2} \Rightarrow \frac{d^2\overrightarrow{OO_2}}{dt} = \vec{0}\ ;$$

and

$$\vec{\Omega}_2^0 = \vec{\Omega}_2^1 + \vec{\Omega}_1^0 = \dot{\theta}.\vec{y}_2 + \dot{\psi}.\vec{z}_1 = \begin{cases} -\dot{\psi}\sin\theta \\ \dot{\theta} \\ \psi\cos\theta \end{cases}_{R_2}$$



$$\vec{V}^0(O_2) = \begin{cases} -\dot{\psi}\sin\theta \\ \dot{\theta} \\ \psi\cos\theta \end{cases}_{R_2} \wedge \begin{cases} 0 \\ 0 = \\ L \end{cases}_{R_2} \begin{cases} L\dot{\theta} \\ L\psi\sin\theta \\ 0 \end{cases}_{R_2}$$

4. Velocity of Point A with Respect to R_0 Expressed in Frame R_2

By the solid's kinematics, we write: $\vec{V}^0(A) = \vec{V}^0(O_2) + \vec{\Omega}_3^0 \wedge \overrightarrow{O_2A}$

Point A is in frame R_3 with coordinates: $\overrightarrow{O_2A} = \begin{cases} a \\ 0 \\ 0 \end{cases}_{R_3} = \begin{cases} a\cos\varphi \\ a\sin\varphi \\ 0 \end{cases}_{R_2}$

Where $\vec{V}^0(A) = \begin{cases} L\dot{\theta} \\ L\psi\sin\theta \\ 0 \end{cases}_{R_2} + \begin{cases} -\dot{\psi}\sin\theta \\ \dot{\theta} \\ \dot{\varphi}+\dot{\psi}\cos\theta \end{cases}_{R_2} \wedge \begin{cases} a\cos\varphi \\ a\sin\varphi \\ 0 \end{cases}_{R_2}$

$$\vec{V}^0(A) = \begin{cases} L\dot{\theta} - a\sin\varphi(\dot{\varphi}+\dot{\psi}\cos\theta) \\ L\dot{\psi}\sin\theta + a\cos\varphi(\dot{\varphi}+\dot{\psi}\cos\theta) \\ -a(\dot{\psi}\sin\theta\sin\theta + \dot{\theta}\cos\varphi) \end{cases}_{R_2}$$

5. Acceleration by Differentiation and by the Solid's Kinematics in Frame R2R_2R2

5.1. **By Differentiation**

We know: $\theta\dot{\psi}$ = Cte ; $\dot{\theta}$ = Cte ; $\dot{\varphi}$ = Cte.

$$\vec{\gamma}^0(O_2) = \frac{d^0\vec{V}^0(O_2)}{dt} = \frac{d^2\vec{V}^0(O_2)}{dt} + \vec{\Omega}_2^0 \wedge \vec{V}^0(O_2)$$

This gives:

$$\vec{\gamma}^0(O_2) = \begin{cases} 0 \\ L\dot{\psi}\dot{\theta}\cos\theta \\ 0 \end{cases}_{R_2} + \begin{cases} -\dot{\psi}\sin\theta \\ \dot{\theta} \\ \dot{\varphi}+\dot{\psi}\cos\theta \end{cases}_{R_2} \wedge \begin{cases} L\dot{\theta} \\ L\dot{\psi}\sin\theta \\ 0 \end{cases}_{R_2} = \begin{cases} -L\dot{\psi}^2\sin\theta\cos\theta \\ 2L\dot{\psi}\dot{\theta}\cos\theta \\ -L\dot{\theta}^2 - L\dot{\psi}^2\sin^2\theta \end{cases}_{R_2}$$

5.2. **By the Solid's Kinematics**

$$\vec{\gamma}^0(O_2) = \vec{\gamma}^0(O) + \frac{d^0\vec{\Omega}_2^0}{dt} \wedge \overrightarrow{OO_2} + \vec{\Omega}_2^0 \wedge (\vec{\Omega}_2^0 \wedge \overrightarrow{OO_2})$$

Points O and O_2belong to the rod; their velocities and accelerations are zero in the frame R_2 attached to the rod:

$\vec{\gamma}^0(O) = \vec{0}$ Because the point O is fixed in the rod

$$\frac{d^0\vec{\Omega}_2^0}{dt}\wedge\overrightarrow{OO_2}=\left\{\begin{matrix}-\dot{\psi}\dot{\theta}\cos\theta\\ 0\\ -\dot{\psi}\dot{\theta}\sin\theta\end{matrix}\right._{R_2}\wedge\left\{\begin{matrix}0\\ 0\\ L\end{matrix}\right._{R_2}=\left\{\begin{matrix}0\\ L\dot{\psi}\dot{\theta}\cos\theta\\ 0\end{matrix}\right._{R_2}$$

$$\vec{\Omega}_2^0\wedge(\vec{\Omega}_2^0\wedge\overrightarrow{OO_2})=\left\{\begin{matrix}-L\dot{\psi}^2\sin\theta\cos\theta\\ L\dot{\psi}\dot{\theta}\cos\theta\\ -L\dot{\theta}^2-L\dot{\psi}^2\sin^2\theta\end{matrix}\right._{R_2}$$

Summing these three expressions gives:

$$\vec{\gamma}^0(O_2)=\left\{\begin{matrix}0\\ L\dot{\psi}\dot{\theta}\cos\theta\\ 0\end{matrix}\right._{R_2}+\left\{\begin{matrix}-L\dot{\psi}^2\sin\theta\cos\theta\\ L\dot{\psi}\dot{\theta}\cos\theta\\ -L\dot{\theta}^2-L\dot{\psi}^2\sin^2\theta\end{matrix}\right._{R_2}=\left\{\begin{matrix}-L\dot{\psi}^2\sin\theta\cos\theta\\ 2L\dot{\psi}\dot{\theta}\cos\theta\\ -L\dot{\theta}^2-L\dot{\psi}^2\sin^2\theta\end{matrix}\right._{R_2}$$

Chapter V

Dynamics of Rigid Solids

Chapter V: Dynamics of the Rigid Body

V.1 Introduction

Dynamics allows us to analyze the links between the movements described by kinematics and the forces or actions that cause them. It examines the concept of force and, more broadly, the concept of efforts exerted on any material system.

The purpose of this chapter is to state the fundamental principle of dynamics and its influence on the study of motion. We will also introduce the concept of the wrench of external forces, which is necessary for writing the fundamental principle of dynamics.

V.2 Expression of the Fundamental Law of Dynamics

Consider a material system (S) that is not isolated, subjected to interactions in a Galilean reference frame $R_0(O, \vec{x}_0, \vec{y}_0, \vec{z}_0)$. For such a system, two types of actions are identified:

- Internal Mechanical Actions:
These are the result of one part of (S) acting on another part of (S). These forces are called internal forces and are denoted as $d\vec{F}_i$.

- External Mechanical Actions:

These stem from the interaction of the rest of the universe (the external environment) with (S). These forces are called external forces and are denoted as $d\vec{F}_e$.

The proper classification of forces as internal or external depends on appropriately selecting the boundary conditions of the system.

At any arbitrary point M within the system (S), the fundamental relationship of dynamics is expressed as:

$$d\vec{F}_i + d\vec{F}_e = \vec{\gamma}(M)dm$$

dm: represents the infinitesimal mass element at M;

$\vec{\gamma}(M)$: is the acceleration vector at M.

Summing over the entire material system gives:

$$\int_S d\vec{F}_i + \int_S d\vec{F}_e = \int_S \vec{\gamma}(M)dm$$

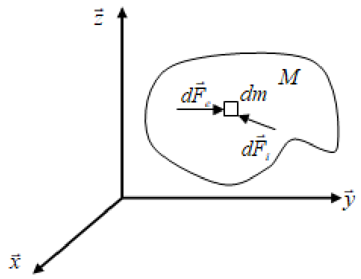


Figure V.1: Mechanical Actions

At any point A in space, the moments of these forces are given by:

$$\int_S \overrightarrow{AP} \wedge d\vec{F}_i + \int_S \overrightarrow{AP} \wedge d\vec{F}_e = \int_S \overrightarrow{AP} \wedge \vec{\gamma}(M)dm$$

We assume that the material system (S) does not exchange matter with other systems and that its total mass is constant.

The external mechanical actions acting on (S) are represented by the torsor $[\tau]_{Fext/A}$, called the external forces wrench, whose components at point A are:

$$[\tau]_{FextA} = \begin{cases} \vec{F}_{ext} \\ \vec{M}_{Aext} \end{cases}$$

$\vec{F}_{ext}$: The resultant of the external forces acting on the system (S);

$\vec{M}_{Aext}$: the moment at point A of the external forces acting on the system (S).

The fundamental principle of dynamics shows that in any Galilean reference frame, the dynamic wrench $[D]_A$ of system (S) is equal to the external forces wrench $[\tau]_{Fext/A}$ calculated at the same point A.

The components of the dynamic wrench $[D]_A$ of system (S) in the Galilean reference frame $R_0(O,\vec{x}_0,\vec{y}_0,\vec{z}_0)$ are: $[D]_A = \begin{cases} \vec{D} \\ \vec{\delta}_A \end{cases}$

$\vec{D}$: The dynamic resultant;

$\vec{\delta}_A$: The dynamic moment at point A.

The equality of the two wrenches implies the equality of their components. This principle generalizes Newton's laws. The components of the two wrenches can be calculated separately, and the obtained expressions are then equated.

Point A, with respect to which the moments are calculated, is arbitrary, but its selection should facilitate the writing of equations. Often in mechanics problems, the center of mass of the system is chosen because the moment of inertia involved in the calculations is easier to determine.

V.2.1 Theorem of Dynamic Resultant

Consider a material system (S) in motion in a Galilean reference frame $R_0(O,\vec{x}_0,\vec{y}_0,\vec{z}_0)$ and subjected to external actions. The dynamic resultant of the material system (S) is equal to the resultant of the external mechanical actions (forces).

$$\vec{D}(S/R_0) = m\vec{\gamma}(G/R_0) = \sum \vec{F}_{ext}$$

G: the center of mass of the system. The resultant of the external forces is equal to the mass of the system times the acceleration of its center of mass.

V.2.2 Theorem of Dynamic Moment

Consider a material system (S) in motion in a Galilean reference frame $R_0(O,\vec{x}_0,\vec{y}_0,\vec{z}_0)$ and subjected to external actions. The dynamic moment of the material system (S) at any point A is equal to the moment of the external mechanical actions (forces) at the same point A.

$$\vec{\delta}_A(S/R_0) = \vec{M}_A(S/R_0)$$

At the center of mass of the system, this equality can be written as:

$$\vec{\delta}_G(S/R_0) = \vec{M}_G(S/R_0) = \frac{d\vec{\sigma}_G(S/R_0)}{dt}$$

As previously demonstrated, the angular momentum at point G, the center of mass of the system, is independent of the reference frame in which it is measured. Therefore, it is often simpler to calculate the dynamic moments at the center of mass of the systems.

Remark:

The dynamic moment of a composite system is equal to the sum of the dynamic moments of its components with respect to the same point.

V.2.3 Scalar Equations Derived from the Fundamental Principle

The vector equations of the dynamic resultant and dynamic moment each lead to three scalar equations, giving a total of six scalar equations for a given material system.

The choice of the reference point for expressing the equation of the dynamic resultant and the point where the dynamic moment is calculated must be judicious to simplify the mathematical writing of scalar equations.

These scalar equations are second-order differential equations and are generally nonlinear. They include the system's inertia characteristics, geometric data, and the components of the mechanical actions applied to the system.

V.3 Principle of Action and Reaction

Two arbitrary points A and B in a material system (S) interact, mutually influencing each other through actions and reactions (Fig. V.2):

$\vec{F}_{A/B}$: Action of A on B.

$\vec{F}_{B/A}$: Action of B on A.

These two actions balance each other. The principle of action and reaction is expressed as: $\vec{F}_{A/B} + \vec{F}_{B/A} = \vec{0}$

This equation implies that the forces are collinear along the line joining the two points A and B, such that: $\vec{F}_{A/B} = \lambda\overrightarrow{AB}$ and $\vec{F}_{B/A} = \lambda\overrightarrow{BA}$

$$\vec{F}_{A/B} + \vec{F}_{B/A} = \lambda\overrightarrow{AB} + \lambda\overrightarrow{BA} = \lambda(\overrightarrow{AB} - \overrightarrow{AB}) = \vec{0}$$

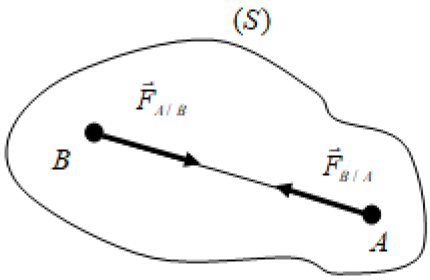


Figure V.2: Action and Reaction

V.3.1 Theorem of Action and Reaction

Consider two material systems (S_1) and (S_2) moving in a Galilean reference frame R_0. Let (S) be the system formed by the union of the two systems: (S) = (S_1) $\cup$ (S_2).

The Toeror of the external forces acting on (S) is decomposed as:

$[\tau]_{Fext_1}$: Resultant of the external actions of the environment on S.

$[\tau]_{12}$: Resultant of the actions of S_2 on S_1.

The torsor of the external forces acting on S_2 is decomposed as:

$[\tau]_{Fext_2}$: Resultant of the external actions of the environment on S_2.

$[\tau]_{21}$: Resultant of the actions of S_1 on S_2.

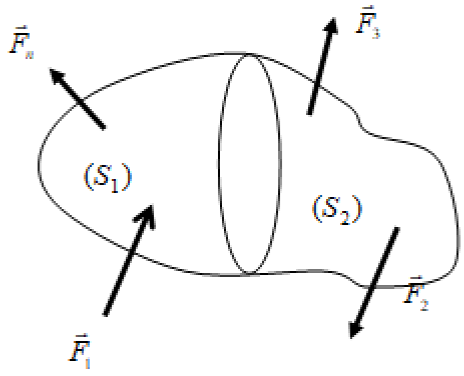


Figure V.3: Resultant of actions

Applying the fundamental principle of dynamics in the Galilean reference frame R_0 to the different systems:

- A (S_1) : $[D]_1 = [\tau]_{Fext1} + [\tau]_{12}$
- A (S_2) : $[D]_2 = [\tau]_{Fext2} + [\tau]_{12}$
- A (S) : $[D] = [\tau]_{Fext1} + [\tau]_{Fext2}$

Knowing that: $[D] = [D]_1 + [D]_2$

The expression represents the theorem of action and reaction.

$$[\tau]_{Fext1} + [\tau]_{Fext2} = [\tau]_{Fext1} + [\tau]_{21} + [\tau]_{Fext2} + [\tau]_{12} \Leftrightarrow [\tau]_{21} + [\tau]_{12} = [0] \Leftrightarrow [\tau]_{21} = -[\tau]_{12}$$

This expression reflects the theorem of action and reaction. For the material system (S), the relation: $[\tau]_{21} + [\tau]_{12} = [0]$ characterizes the internal actions.

In general, when all the internal mechanical actions within a material system (S) can be represented by a wrench $[\tau]_{\text{int F}}$, it is always zero.

V.3.2 Properties of Internal Forces

The torsor of internal forces has the following components: $[\tau]_{F\,\text{int}} = \begin{cases} \vec{R}_{\text{int}} = \vec{0} \\ \vec{M}_{A\,\text{int}} = \vec{0} \end{cases}$

$$\vec{R}_{\text{int}} = \sum_{i=1} (\vec{F}_{ij} + \vec{F}_{ji}) = \vec{0}$$

Action-Reaction Forces: $\vec{F}_{ij} = -\vec{F}_{ji}$

Moment of Internal Forces: At any point A in space, the moment of internal forces is given by:

$$\vec{M}_{A\,\text{int}} = \sum_i (\overrightarrow{AM_i} \wedge \vec{F}_{ij} + \overrightarrow{AM_j} \wedge \vec{F}_{ji}) = \sum_i (\overrightarrow{AM_i} \wedge \vec{F}_{ij} + (\overrightarrow{AM_i} + \overrightarrow{M_iM_j}) \wedge \vec{F}_{ji})$$

$$\vec{M}_{A\text{int}} = \sum_i (\overrightarrow{AM_i} \wedge (\vec{F}_{ij} + \vec{F}_{ji}) + (\overrightarrow{M_i M_j}) \wedge \vec{F}_{ji}) = \vec{0}$$

Because $(\vec{F}_{ij} + \vec{F}_{ji}) = \vec{0}$ et $(\overrightarrow{M_i M_j}) \wedge \vec{F}_{ji} = \vec{0}$

The torsor of internal forces is always a null wrench: $[\tau]_{F\ int} = 0$

V.4 Kinetic Energy Theorem

In many cases, determining the equation of motion for a rigid body or a system of rigid bodies is easier using the kinetic energy theorem, which helps simplify the solution to mechanical problems.

Furthermore, the derivative of kinetic energy is related to the power of both internal and external forces acting on the body.

V.4.1 Work and Power of a Force

Consider a discrete system composed of n particles M_i of mass m_i, moving in a Galilean reference frame $R(\vec{x}, \vec{y}, \vec{z})$. Let $\overrightarrow{OM_i}$ be the position vector of the particle M_i in the reference frame R. Its velocity vector is:

$$\vec{V}(M_i) = \frac{d\overrightarrow{OM_i}}{dt} \Rightarrow d\overrightarrow{OM_i} = \vec{V}(M_i)dt$$

$d\overrightarrow{OM_i}$: represent the infinitesimal displacement during a time dt.

If the particle M_i is subjected to a force $\vec{F}$, the infinitesimal work done by this force is:

$$dW_i = \vec{F}_i . d\overrightarrow{OM_i}$$

The power received by the particle is :

$$P_i = \frac{dW_i}{dt} = \vec{F}\,\frac{d\overrightarrow{OM_i}}{dt}\,\vec{F}.\vec{V}(M_i)$$

Note that $\vec{F}_i$ includes both internal forces $\vec{F}_{i\text{int}}$ and external forces $\vec{F}_{iext}$:

$\vec{F}_i = \vec{F}_{i\text{int}} + \vec{F}_{iext}$; For the entire system, the total power is:

$$W = \sum_i \vec{F}_i . d\overrightarrow{OM_i} = \sum_i (\vec{F}_{i\text{int}} + \vec{F}_{iext}) . d\overrightarrow{OM_i}$$

$$P = \sum_i \vec{F}_i . \vec{V}(M_i) = \sum_i (\vec{F}_{i\text{int}} + \vec{F}_{iext}) . \vec{V}(M_i)$$

V.4.2 Kinetic Energy Theorem

For a system of n particles M_i with mass m_i and velocity $\vec{V}(M_i)$, moving in a Galilean reference frame $R(\vec{x},\vec{y},\vec{z})$, the kinetic energy is:

$$E_c = \sum_{i=1}^{n} \frac{1}{2} m_i \left(\vec{V}(M_i)\right)^2$$

The time derivative of this expression is:

$$\frac{dE_c}{dt} = \sum_{i=1}^{n} \frac{\vec{V}(M_i)}{dt} . \vec{V}(M_i)$$

The force acting on particle M_i: $\vec{F_i} = m_i \dfrac{d\vec{V}(M_i)}{dt}$, $\quad \dfrac{dE_c}{dt} = \sum_{i=1}^{n} \vec{F_i} .. \vec{V}(M_i) = P$

Since $\vec{F_i}$ includes both internal and external forces, this can be written as:

$$\frac{dE_c}{dt} = P_{\text{int}} + P_{ext}$$

P_{int} : power from internal forces;

P_{ext} : power from external forces.

The power of the internal and external forces equals the time derivative of kinetic energy. Integrating this expression between two instants t_1 and t_2, the kinetic energy theorem becomes:

$$E_c(t_2) - E_c(t_1) = \int_{t_1}^{t_2} (P_{\text{int}} + P_{ext}) dt$$

$$E_c(t_2) - E_c(t_1) = W_{\text{int}} - W_{ext}$$

The variation in kinetic energy between two instants t_1 and t_2 equals the work of all internal and external forces applied to the system.

V.4.3 Kinetic Energy of a Rigid Body

For a rigid body, the kinetic energy is given by:

$$E_c = \frac{1}{2} \int_s \vec{V}^2(M) dm$$

Let $R_0(O,\vec{x}_0,\vec{y}_0,\vec{z}_0)$ be a fixed orthonormal frame and $R_1(O,\vec{x}_1,\vec{y}_1,\vec{z}_1)$ a frame attached to the rigid body (S), moving in any manner such that $O_1 \in (S)$

Let $\vec{\Omega}_1^0$ be the angular velocity of frame R_1 relative to frame R_2, and M an arbitrary point of the solid. According to the kinematics of the solid, we write:

$$\vec{V}^0(M) = \vec{V}^1(M) + \vec{\Omega}_1^0 \wedge \overrightarrow{O_1M}$$

The kinetic energy of the solid (S) in motion relative to a fixed frame R_0 is expressed as:

$$\frac{dE_c^0}{dt} = \int_s \vec{V}^0(M).\frac{\vec{V}^0(M)}{dt}dm = \int_s \vec{V}^0(M).\vec{\gamma}^0(M)dm$$

$$\frac{dE_c^0}{dt} = \int_s (\vec{V}^0(O_1) + \vec{\Omega}_1^0 \wedge \overrightarrow{O_1M}).\vec{\gamma}^0(M)dm$$

Using the permutation rule in the scalar triple product, the expression of V_{ent}:

$$\frac{dE_c^0}{dt} = \vec{V}^0(O_1).\int_s \vec{\gamma}^0(M)dm + \vec{\Omega}_1^0.\int_s \overrightarrow{O_1M} \wedge \vec{\gamma}^0(M)dm$$

Which can also be written in the form of the product of two torsor:

$$\frac{dE_c^0}{dt} = \begin{Bmatrix} \vec{\Omega}_1^0 \\ \vec{V}^0(O_1) \end{Bmatrix} . \begin{Bmatrix} \int_s \vec{\gamma}^0(M)dm \\ \int_s \overrightarrow{O_1M} \wedge \vec{\gamma}^0(M)dm \end{Bmatrix} = [C_{O_1}][D_{O_1}]$$

The derivative of kinetic energy is equal to the product of the kinematic and dynamic wrenches, and thus is equal to the power of the absolute acceleration quantities.

As we saw earlier, according to the fundamental theorem of dynamics, the dynamic wrench is equal to the wrench of external forces for a rigid body, hence the final expression:

$$\frac{dE_c}{dt} = P_{ext}$$

V.4.4 Conservation of Total Energy

The kinetic energy theorem can be written as:

$$dE_c = P_{ext}dt = dW_{ext}$$

If all external forces derive from a potential function U(r), independent of time, then: $\vec{F}_{ext} = -\overrightarrow{grad}U(r)$ From this, we deduce:

$$dW_{ext} = \vec{F}_{ext}.d\vec{r} = -dU(r)$$

The kinetic energy theorem becomes:

$dE_c = -dU(r) \Rightarrow d(E_c + U) = 0$ and finely $E_c + U = Cte$

$E_C + U = E$, E : total energy

This expression represents the total energy conservation theorem:

Application Exercises

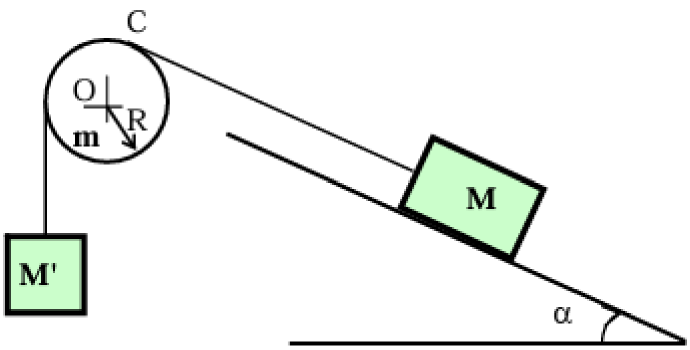
Exercise 1

A system consists of two masses M and M′ connected by an inextensible cable that passes over a pulley of radius R. The mass M′ is suspended vertically, and the mass M slides without friction on an inclined plane at an angle α\alphaα. The friction of the cable on the pulley is negligible. Write:

1. The relation between the pulley's angular velocity $\vec{\Omega}$ and the acceleration $\vec{\gamma}$ of the two bodies.
2. The fundamental principle of dynamics and determine the system's acceleration in two cases:

    a) The pulley's mass is negligible.

    b) The pulley's mass is m.



Exercise 2

A homogeneous bar of length AB=L, mass m, and center G, has one end A resting on a smooth horizontal surface, and the other end B sliding along a vertical wall. Initially, the bar makes an angle θ_0 with the wall. Both ends slide without friction.

1. Using the theorems of dynamic resultant and dynamic moment, establish the three scalar equations of the bar's motion.
2. Deduce the angular acceleration $\ddot{\theta}$ from these equations.
3. Show, by integrating the acceleration equation, that: $\theta^2 = \frac{3g}{L}(\cos\theta_0 - \cos\theta)$
4. Re-derive the above expression $\dot{\theta}^2$ using the total mechanical energy conservation theorem.
5. Determine R_A and R_B (the reactions at A and B) as functions of θ.

6. Find the angle at which the bar detaches from the wall.