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Course Notes for Maths 2

For First Year Students S.T and S.M

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Dedications

I dedicate this humble work:

To the most precious person in my life, my mother.

To the one who made me who I am, my father.

I also dedicate it to my dear wife and my children, to my beloved brothers and sisters, and to all my friends and colleagues.

Contents

Introduction

This handout contains the mathematics II course that I teach in the second semester of the first year of science and technology. I want to point out the principal mathematics tools of algebra and analysis that a student must assimilate and learn. That is, this document can be used as a reference text for undergraduates in the first year in Science and Technology who will be facing mathematics problems and will be interested in learning techniques to solve them.

The course is divided into five chapters:

Chapter 1: We defined the integral definite and finite, we give properties, and it's application.

Chapter 2: We give the application the integral in the differential equations of the first and second order.

Chapter 3: Contains the notion in matrix.

Chapter 4: We applicated the matrices for solve the linear systems.

Chapter 5: Finaly we study the continuous and derivative, double and triple integral for multivariable functions.

Chapter 1

The Integral

1.1 The indefinite integral

Definition 1.1.1.
A function F is called an ***antiderivative*** *of f on I*
iff: $\forall x \in I, F^{'}(x) = f(x)$ and denoted by: $\displaystyle\int f(x)dx$.

Example 1.1.1.

- $F_1(x) = x^2$ *is an* ***antiderivative*** *of $f(x) = 2x$ on $\mathbb{R}$.*

- $F_2(x) = x^2 + 10$ *is also an* ***antiderivative*** *of $f(x) = 2x$ on $\mathbb{R}$.*

Theorem 1.1.1.
If F is ***antiderivative*** *of f, then:*
every function $G(x) = F(x) + c$,($c \in \mathbb{R}$) is an ***antiderivative*** *of f and write:*

$$\int \cos(x)dx = \sin(x) + c, \quad c \in \mathbb{R}.$$

Example 1.1.2.
Since $\sin$ *is an* ***antiderivative*** *of* $\cos$*, then:*

$$\int f(x)dx = F(x) + c$$

Properties 1.1.1.
Just as we had a list of properties the derivatives of sum and products of functions
Let f and g continuous on I, $\lambda \in \mathbb{R}$ (constant)

1. $\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx.$

2. $\int \lambda f(x)dx = \lambda \cdot \int f(x)dx.$

1.1.1 The table of an antiderivatives functions

Function of f		$\int f(x)dx$	**Interval**
1)	$a,\ \ a \in \mathbb{R}$	ax+b	$\mathbb{R}$
2)	$x^n,\ n \neq -1$	$\frac{x^{n+1}}{n+1}+c$	$\mathbb{R}$ or $\mathbb{R}_+^*$
3)	$\frac{1}{x}$	$\ln\|x\|$	$\mathbb{R}^*$
4)	e^x	e^x+c	$\mathbb{R}$
5)	$a^x,\ a>0$	$\frac{a^x}{\ln(a)}+c$	$\mathbb{R}$
6)	$\cos(x)$	$\sin(x)+c$	$\mathbb{R}$
7)	$\sin(x)$	$-\cos(x)+c$	$\mathbb{R}$
8)	$\sinh(x)$	$\cosh x + C$	$\mathbb{R}$
9)	$\cosh(x)$	$\sinh(x)+c$	$\mathbb{R}$
10)	$\frac{1}{1+x^2}$	$\arctan(x)+c$	$\mathbb{R}$
11)	$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x)+c$	$]-1,1[$
12)	$-\frac{1}{\sqrt{1-x^2}}$	$\arccos(x)+c$	$]-1,1[$

1.1.2 The integral of polynomials

We have:

$$\begin{aligned} I &= \int a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 dx, \quad a_n \neq 0 \\ &= \frac{a_n}{n+1}x^{n+1} + \frac{a_{n-1}}{n}x^n + \cdots + a_0 x + c. \end{aligned}$$

Examples 1.1.1.

1) $\int x^3 dx = \frac{1}{4}x^4 + c.$

$\int (x^2+2x)dx = \frac{1}{3}x^3 + x^2 + c,$ *where* c *is real constant.*

1.1.3 The antiderivatives of the compositives functions

Let U is continuous and differentiable on I and c is real constant.

$f(x)$	$u' \cdot u^n,\ (n \neq -1)$	$\frac{u'}{u}$	$u'e^u$	$\frac{u'}{\sqrt{u}}$	$u'\cos(u)$	$u'\sinh(u)$
$\int f(x)dx$	$\frac{u^{n+1}}{n+1}+c$	$\ln\|u\|+c$	e^u+c	$2\sqrt{u}+c$	$\sin(u)+c$	$\cosh(u)+c$

Examples 1.1.2.

1. $\int \tan x dx = \int \frac{sinx}{cosx} dx = \ln|\cos x| + c.$
2. $\int (2x-1)(x^2-x+4)^{20} dx = \frac{1}{21}(x^2-x+4)^{21} + c.$
3. $\int (x^2+1)\sinh(\frac{1}{3}x^3+x-5)dx = cosh(\frac{1}{3}x^3+x-5)+c$, *where c is real constant.*

1.1.4 Method of substitution

Let u is differentiable on I and $F(x) = \int f(x)dx$. Then:

$$\int u^{'} F^{'}(u) = F(u) + c$$

Examples 1.1.3.

- $I_1 = \int \frac{2x+1}{x^2+x+3} dx = lnu + c$, *such that* $u = x^2+x+3 > 0.$
- $I_2 = \int (4x+6)e^{x^2+3x} dx = 2e^{x^2+3x} + c$
- $I_3 = \int \frac{e^x}{e^x+4} = ln(e^x+4) + c.$

1.1.5 Integration by part

Theorem 1.1.2. *If: U, V are differentiable on I.*

$$\int U(x) \cdot V^{'}(x)dx = U(x) \cdot V(x) - \int (U^{'}V)(x)dx$$

Proof:

We have: $(U(x) \cdot V(x))^{'} = U^{'}(x) \cdot V(x) + U(x) \cdot V^{'}(x)$, then
$(U(x) \cdot V(x))^{'} - U(x) \cdot V^{'}(x) = U^{'}(x) \cdot V(x)$, so

$$\int U(x) \cdot V^{'}(x)dx = U(x) \cdot V(x) - \int (U^{'}V)(x)dx$$

Examples 1.1.4.

1. *$I_1 = \int xe^x dx$*
   We put: $U = x, V' = e^x$. Then $I_1 = xe^x - e^x + c$.

2. *$I_2 = \int x\sin(x)dx$*
   Posons: $U = x, V' = sinx$, then $I_2 = -xcosx + sinx + c$.

3. *$\int \ln x = x\ln x - \ln x + c$*

Table: Gives some recommended choices for integration by parts.

Integral	**Choice of U**	**Choice of V**
1) $\int x^n \cos(kx)dx$	$U = x^n$	$V' = \cos(kx)$
2) $\int x^n \sin(kx)dx$	$U = x^n$	$V' = \sin(kx)$
3) $\int x^n e^{kx}dx$	$U = x^n$	$V' = e^{kx}$
4) $\int x^n \ln(kx)dx$	$U = \ln(kx)$	$V' = x^n$

1.1.6 Integrals Trigonometric functions

We know that:

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x)$$

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x)$$

Integral of type:

$$I = \int \sin x^n \cdot \cos x^n dx \quad n \in \mathbb{N}, \quad m \in \mathbb{N}$$

Posons: $t = \sin(x) \Rightarrow \cos(x) = \sqrt{1-t^2}$ $\cos(x) \geq 0$,
by substitution:

Then: $dt = \sqrt{1-t^2}dx$, so:

$$I = \int t^m \cdot (1-t^2)^{\frac{n-1}{2}} dt$$

Examples 1.1.5.

1. *$\int \sin^2(x) \cdot \cos^3(x)dx = ?$*

$$\begin{aligned}\int \sin^2(x)\cdot\cos^3(x)dx &= \int t^2\cdot(1-t^2)^{\frac{3-1}{2}} \\ &= \frac{t^3}{3}-\frac{t^5}{5}+c \\ &= \frac{\sin^3(x)}{3}-\frac{\sin^5(x)}{5}+c.\end{aligned}$$

2. Find $\int \sin(x)\cdot\cos^2(3x)dx$

1.1.7 Integrals Exponentials functions

The type $I=\int f(e^x)dx$
We suppose that: $t=e^x \Rightarrow dt=e^x dx$

$$I=\int \frac{1}{t}f(t)dt$$

Example 1.1.3.
$\int \frac{e^x}{e^x+2}dx=\int \frac{1}{t}(\frac{t}{t+2})dt=ln\mid t+2\mid +c=ln(e^x+2)+c.$

1.1.8 Integrals Rational functions

A rational function is a quotient of two polynomials: $R(x)=\frac{P(x)}{Q(x)}$.

$$I=\int \frac{P(x)}{Q(x)}$$

If: $deg\ p \geq deg\ Q$, use long division, then write:

$$R(x)=R(x)+\frac{P_1(x)}{Q_1(x)},\quad deg\ P_1<deg\ Q_1.$$

Decompose $\frac{P_1(x)}{Q_1(x)}$ into partial fractions of the form:

$$\frac{P_1(x)}{Q_1(x)}=F_1(x)+F_2(x)+\cdots$$

where: each fraction is of the form:

$$F_2(x)=\frac{A}{(x+b)^n}\quad or\quad \frac{Ax+B}{(x^2+bx+c)^k}\ \ 1\leq n$$

Calculate partial fractions

- a) $I_1 = \displaystyle\int \frac{A}{(x+b)^n}dx$, $n \in \mathbb{R}^*$. If $n = 1$: $I_1 = A\ln|x+b| + c$.
  $If n \neq 1$: $I_1 = \frac{A}{-n+1}(x+b)^{-n+1} + c$

- b) $I_2 = \frac{Ax+B}{(x^2+bx+c)^n}$
  If $n = 1$:

$$\begin{aligned} I_2 &= \int \frac{Ax+B}{(x^2+bx+c)} \\ &= A_1 \int \frac{u'}{u} + B_1 \int \frac{1}{u^2-a^2} \quad or \quad = A_1 \int \frac{u'}{u} + B_1 \int \frac{1}{u^2+a^2} \\ &= \lambda \ln|x^2+bx+c| + \mu \arctan(u) \quad or \quad = \lambda \ln|x^2+bx+c| + \mu \ln|u(x)|. \end{aligned}$$

  If $n \neq 1$: $I_2 = A_1 \displaystyle\int \frac{u'}{u} + B_1 \int \frac{1}{(x^2+bx+c})^n$ (By induction)

Examples 1.1.6.

1. $I_1 = \displaystyle\int \frac{3}{(x-1)^4}dx = -(x-1)^{-3} + c.$
2. $I_2 = \displaystyle\int \frac{1}{4+x^2}dx = \frac{1}{2}\arctan(\frac{x}{2}).$
3. $I_3 = \displaystyle\int \frac{1}{\sqrt{9-x^2}}dx = \arcsin(\frac{x}{3}) + c$
4. $I_4 = \displaystyle\int \frac{x+1}{x^2-4x+80}dx$

$$\begin{aligned} I_4 &= \int \frac{1}{2}\left(\frac{2x+2}{(x^2-4x+8)}dx\right) \\ &= \frac{1}{2}\left[\int \frac{2x-4}{x^2-4x+8} + \frac{6}{x^2-4x+8}dx\right] \\ &= \frac{1}{2}\ln|x^2-4x+8| + 3\int \frac{3}{(x-2)^2+4} \\ &= \frac{1}{2}\ln|x^2-4x+8| + \frac{3}{2}\arctan(\frac{x-2}{2}) + c. \end{aligned}$$

Results:

1. $\displaystyle\int \frac{u'}{1+u^2}dx = \arctan(u) + c.$

2. $\displaystyle\int \frac{1}{a^2+u^2}dx = \frac{1}{a}\arctan(u) + c.$

3. $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin(\frac{x}{a}) + c.$

4. $\displaystyle\int \frac{u'}{\sqrt{1-u^2}}dx = \arcsin(u) + c.$

Example 1.1.4.
$\displaystyle\int \frac{2x-7}{x^2+x+4}dx = \int \left(\frac{2x+1}{x^2+x+4} - \frac{8}{x^2+x+4}dx\right)$

1.2 The Definite integral

Definition 1.2.1.
Let f is continuous on $[a,b]$ and F is an antiderivative of f. The definite integral of f from a to be is:

$$\int_a^b f(x)dx = F(a) - F(b) = [F(x)]_a^b$$

Examples 1.2.1.
$\displaystyle\int_0^{\frac{\pi}{2}} \cos(2x)dx = \frac{1}{2}[sin2x]_0^{\frac{\pi}{2}} = -\frac{1}{2}.$

Remarks 1.2.1.

1. *If $f(x) > 0$ on $[a,b]$. Then:*

$$\int_a^b f(x)dx = \mathcal{A}(S)$$

2. $\displaystyle\int_a^b f(x)dx = \int_a^b f(t)dt$

Properties 1.2.1.
Let f, g are continuous on $[a,b]$, $\lambda \in \mathbb{R}$

1. $\displaystyle\int_a^b \lambda f(x)dx = \lambda \cdot \int_a^b f(x)dx.$

2. $\displaystyle\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$

3. $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx, \ \forall a, b, c$
   En particular: $\int_a^a f(x)dx = 0.$

4. *If* $\forall x \in [a, b], f(x) \geq 0$. *Then:* $\int_a^b f(x)dx \geq 0$

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx$$

5. *If* $\forall x \in [a, b], f(x) \leq g(x)$. *Then:*

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

Examples 1.2.2.

1. $\int_{-1}^2 |x-1|dx = \int_{-1}^1 -(x-1)dx + \int_1^2 (x-1)dx$
2. $\int_0^\pi x\cos(x)dx$ *(by parts)*

1.2.1 Improper Integrals

Definition 1.2.2.
Improper integrals are integrals in which one a booth the forms:

$$\int_a^{+\infty} f(x)dx, \ \int_{-\infty}^a f(x)dx, \ \int_{-\infty}^{+\infty} f(x)dx, \ \int_a^b f(x)dx,$$

where f is not continuous on interval.

- $\int_a^{+\infty} f(x)dx = \lim_{b \to +\infty} \int_a^b f(x)dx$ (converge or diverge)
- $\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^{+\infty} f(x)dx.$

Examples 1.2.3.

1. $\int_0^{+\infty} e^{-x}dx = [-e^{-x}]_0^{+\infty} = 1.$

2. $\int_{-\infty}^{2} \frac{1}{x^2} dx = [-\frac{1}{x}]_{-\infty}^{2} = -\frac{1}{2}.$

3. $\int_{0}^{1} \ln(x) dx = \lim_{a \to 0} \int_{a}^{1} \ln(x) = \lim_{a \to 0} [x \ln(x) - x]_{a}^{1} = -1.$

Chapter 2

Differential Equations

In this section we give a different method for solve a differential equation.

2.1 Ordinary differential equation 3

Definition 2.1.1.
An equation involving the derivatives of an unknown function y of a single variable x over an interval $x \in I$. It can be written in the form:

$$F(x, y, y', ..., y^{(n)}) = 0, \tag{2.1.1}$$

where $n \in \mathbb{N}^*$ is called the order of this equation.

Shortly (2.1.1) is denoted by ODE.

2.1.1 Solution of ODE

Definition 2.1.2. [7]*(The General Solution)*
The Solution of a differential equation containing as many arbitrary constants as the order of the differential equation is called as the general solution.

Example 2.1.1.
Let $\acute{y} = \frac{1}{x}$. The general solution of this equation on $\mathbb{R}^*$ is $f(x) = ln|x| + c$, where $c \in \mathbb{R}$ (constant) .

Definition 2.1.3. ***(Particular Solution)***
The Solution obtain by giving particular values to the arbitrary constants in the general solution is called particular solution.

Example 2.1.2.
$g(x) = ln(x)$ is a particular solution of differential equation $\acute{y} = \frac{1}{x}$ on R_+^*.

Examples 2.1.1.

1. $x + 2y + x^2y' = 0$, *is ODE the first order.*
2. $x - 2 + y'' = x^3$, *is ODE the second order.*

2.2 First Order Differential Equations [3]

Definition 2.2.1.
The general first order ODE has the form $F(x, y, \acute{y}) = 0$ *(i.e* $\acute{y} = f(x, y)$*), where* y *is a function of* x*.*

The Solution of 1^{st} order and 1^{st} degree differential equation is obtained by following methods if they are in some standard forms as i). Variable separable form ii) Linear differential equation form.

2.2.1 Separable Equations

Definition 2.2.2.
The first order ODE is said to be separable if can be expressed as:

$$\acute{y} = \frac{f(x)}{g(y)} i.e : f(x)dx = g(y)dy. \tag{2.2.1}$$

To solve (2.2.1), we have (2.2.1) $\Longrightarrow \int f(x)dx = \int g(y)dy$.

Examples 2.2.1.
Solve the following equations:
1) $\acute{y} = x^2 - 2x$
2) $\frac{\acute{y}}{y} = 2x - 3$

Solution:

1) We have $\acute{y} = x^2 - 2x \Longrightarrow dy = (x^2 - 2x)dx$
Integrating both sides we have:
$\int dy = \int (x^2 - 2x)dx$
$\Longrightarrow y = \frac{1}{3}x^3 - x^2 + c$, (c is constant).
2)For $y \neq 0$, we have
$\frac{\acute{y}}{y} = 2x - 3 \Longrightarrow \frac{1}{y}dy = (2x - 3)dx$
$\Longrightarrow \int \frac{1}{y}dy = \int (2x - 3)dx$
$\Longrightarrow ln|y| = x^2 - 3x + c_1$, ($c_1 \in \mathbb{R}$)
$\Longrightarrow |y| = e^{(x^2-3x)}.k$, such that $k = e^{c_1}$.
Then we can write the general solution in the form $y = c.e^{(x^2-3x)}$, where $c \in R^*$.

2.2.2 Linear ODE of 1st order [2]

Definition 2.2.3.
A differential equation of the form:

$$\acute{y} = a(x)y + b(x), \tag{2.2.2}$$

is called a first-order linear (in y and $\acute{y}$) equation.Where a and b the fuctions continuous on interval I.

Remark 2.2.1.
If $b(x) = 0$, then the equation (2.2.3) is called homogeneous i.e in the form:

$$\acute{y} = a(x)y.$$

(Note that the homogeneous equation is separable.) Otherwise, the equation (2.2.3) is called nonhomogeneous.

A ODE can be solved by the theorem:

Theorem 2.2.1.
Consider the first-order linear equation

$$\acute{y} = a(x)y + b(x). \tag{2.2.3}$$

The general solution is the sum of the homogeneous solution and a particular solution:

$$y = y_h + y_p,$$

where $y_h = c.e^{A(x)}$ (A is a primitive of a), y_p is a particular solution of (2.2.3).

There are two very common methods to solve linear equations of first order. The first is called the method of integrating factors. The second is called variation of parameters.

Example 2.2.1.
Let the ODE: $\acute{y} = y - x + 1$....(*)
1) Show that is $y_p = x$ is a solution of (*).
2) Find the general solution of (*).

Solution:
1) We have: $\acute{y}_p = 1$. Then $1 = x - x + 1$ is true, so $y_p = x$ is a solution of (*).
2) The general solution of (*):
We know that: the solution of homogeneous equation: $\acute{y} = y$, is $y_h = c.e^x$, $c \in \mathbb{R}$. Then the general solution of (*) is: $y = c.e^x + x$, where $c \in \mathbb{R}$.

2.2.3 The Integration Factor Method

In this section we discuss a technique for solving the first order linear non-homogeneous equation

$$\acute{y} + a(x)y = b(x). \tag{2.2.4}$$

We multiply the both sides of equation (2.2.4) by the function $\mu(x) \neq 0$, , we get

$$\mu(x)\acute{y} + a(x)\mu(x)y = b(x)\mu(x). \tag{2.2.5}$$

We put: $a(x)\mu(x) = \acute{\mu}(x)$, this implie that:

$$\mu(x) = e^{\int (a(x)dx},$$

and from (2.2.5), we deduce that:

$$(\mu(x)y)' = b(x)\mu(x).$$

So,

$$y = \frac{1}{\mu(x)} \int b(x)\mu(x)dx + \frac{c}{\mu(x)},$$

as the general solution of (2.2.4), c is constant real.
$\mu(x)$ called **the integration factor**.

Example 2.2.2.
Solve the following equation

$$\acute{y} + 2xy = x. \tag{2.2.6}$$

We have: $a(x) = 2x, b(x) = x$. *The integrating factor is:* $\mu(x) = e^{\int 2xdx} = e^{x^2}$. *Then*

$$\begin{aligned} y &= e^{-x^2} \int xe^{x^2} dx + ce^{-x^2} \\ &= \frac{1}{2} e^{-x^2}.e^{x^2} + ce^{-x^2} \\ y &= \frac{1}{2} + ce^{-x^2}, c \in \mathbb{R}. \end{aligned}$$

Cauchy problem:
There exist the unique solution satisfying:

$$\begin{cases} \acute{y} = f(x, y) \\ y_0 = f(x_0). \end{cases}$$

Example 2.2.3.
In example 2.2.2, deduce the solution of (2.2.6) *which is satisfying the initial condition:* $y(0) = \frac{3}{2}$.
We have $y = \frac{1}{2} + ce^{-x^2}$, *then* $y(0) = \frac{1}{2} + c$.
Therefore, $y(0) = \frac{3}{2} \Longleftrightarrow c = 1$. *So there exist the unique solution* $y = \frac{1}{2} + e^{-x^2}$.

2.2.4 The constant variation method

In the solution $y = c.e^{A(x)}$ of homogeneous equation $\acute{y} = a(x)y$, we suppose that c is a function of x, such that y is a solution of non-homogeneous (2.2.4) , after this we calculate $c(x)$.

Example 2.2.4.
Solve the equation: $\acute{y} + 2xy = 2xe^{-x^2}$*...(E)*
The homogeneous equation is: $\acute{y} + 2xy = 0$*...(H)*
The solution of (H) is: $y_h = c.e^{-x^2}, c \in \mathbb{R}$.
We suppose that $y = c(x).e^{-x^2}$*, is a solution of (E).*
We have: $\acute{y} = c(\acute{x}).e^{-x^2} - 2xc(x)e^{-x^2}$*, we substitution in (E), we obtained:*
$\acute{c}(x)e^{-x^2} - 2xc(x)e^{-x^2} + 2xe^{-x^2} = 2xe^{-x^2}$*, then* $\acute{c}(x) = 2x$.
So: $c(x) = x^2 + k$*,* $k \in \mathbb{R}$*, therefore the general solution of (E) is:* $y = (x^2 + k)e^{-x^2}$.

2.2.5 The Bernoulli Equation

In 1696 Jacob Bernoulli solved what is now known as the Bernoulli differential equation. This is a first order nonlinear differential equation. The following year Leibniz solved this equation by transforming it into a linear equation. We now explain Leibniz's idea in more detail.

Definition 2.2.4. [6]
The Bernoulli equation is

$$\acute{y} = a(x)y + b(x)y^n, \tag{2.2.7}$$

where a*,* b *are two functions continuous on* I *and* $n \in \mathbb{N}$.

Remarks 2.2.1.

1. *If* $n = 0$ *or* $n = 1$*: the equation* (2.2.7) *is linear.*
2. *If* $n \in \mathbb{N}^* - \{1\}$*: the equation* (2.2.7) *is nonlinear.*

For the solution of (2.2.7) it is a nonlinear equation that can be transformed into a linear equation.
If $n \in \mathbb{N}^* - \{1\}$, we multiply (2.2.7) by y^{-n}, we get

$$\acute{y}y^{-n} = a(x)y^{1-n} + b(x)$$

Introduce the new unknown: $z = y^{1-n}$ and compute it's derivative $\acute{z} = (1 - n)\acute{y}y^{-n}$.
If we substitute, we obtain the linear equation

$$\acute{z} = (1 - n)a(x)z + (1 - n)b(x),$$

we calculate z and after this we deduce y.

Example 2.2.5.
Solve the equation:

$$\acute{y} = y + xy^2. \tag{2.2.8}$$

This equation is Bernoulli's with $n = 2$.
We put: $z = y^{1-2}$ *i.e* $z = \frac{1}{y}, (y \neq 0)$, *then* $\acute{y} = \frac{-\acute{z}}{z^2}$, *we substitute in* (2.2.8)*:*
$\frac{-\acute{z}}{z^2} = \frac{1}{z} + x\frac{1}{z^2}$, *we get*
$\acute{z} = -z - x$ *is a linear ODE. By integration factor,we have:*
$\mu(x) = e^{\int -dx} = e^{-x}$, *and* $z = e^x(\int -xe^{-x}dx) + ce^x, c \in \mathbb{R}$.
Then by part, we deduce that: $z = x - 1 + ce^x$, *so the solutions of* (2.2.8) *are:*
$y = \frac{1}{z} = \frac{1}{x-1+ce^x}$ *or* $y = 0$.

2.3 Second order ODE [4]

Newtons second law:
Consider movement of a point particle along a straight line and let its coordinate at time t be $x(t)$. The velocity (Geschwindigkeit) of the particle is $v(t) = \acute{x}(t)$ and the acceleration (Beschleunigung) is $a(t) = x^{''}(t)$. The Newtons second law says that at any time

$$mx^{''} = F, \tag{2.3.1}$$

where m is the mass of the particle and F is the force (Kraft) acting on the particle. In general, F is a function of $t, x, x^{'}$ so that (2.3.1) can be regarded as a second order ODE for $x(t)$.

Definition 2.3.1.
A general second order ODE, resolved with respect to $y^{''}$ *has the form*

$$y^{''} = f(x, y, y^{'}) \tag{2.3.2}$$

Examples 2.3.1.
1) $y^{''} = l(x)y^{'} - 5 + x^2y^2$.
2) $2xy^{''} + xy^{'} + 5 = 0$

2.3.1 Second Order Linear Equations

Definition 2.3.2.
An operator L *is a linear operator iff for every pair of functions* y_1, y_2 *and constants* c_1, c_2 *holds*

$$L(c_1y_1 + c_2y_2) = c_1L(y_1) + c_2L(y_2).$$

In this Section we work with linear operator $L(y) = ay^{''} + by^{'} + cy$ associated by equation as the following result.

Definition 2.3.3.
A second order linear differential equation for the function y is

$$a(x)y'' + b(x)y' + c(x)y = f(x), \tag{2.3.3}$$

where $a(x) \neq 0, b, c$ and f are continuous on the interval $I \subseteq \mathbb{R}$.

Remarks 2.3.1.

1. (2.3.3) *is homogeneous iff the source: $f(x) = 0$.*

2. (2.3.3) *has constant coefficients iff a, b and c are constants.*

In this section we solve the equations of second order linear with a constant coefficients.

Examples 2.3.2.

1. $y'' + 4y' - 3y = e^x$, *is second ODE linear non-homogeneous with constant coefficients.*

2. $3y'' + xy' + (x^2 - 1)y = 0$, *is second ODE linear homogeneous.*

2.3.2 Solution of the homogeneous equation

Let the homogeneous equation with constant coefficients

$$ay'' + by' + cy = 0, \tag{2.3.4}$$

associated homogeneous of (2.3.3), such that a, b and c are constants and $a \neq 0$.

Theorem 2.3.1. ***(General solution of*** (2.3.4) ***)***
If y_1 and y_2 are linearly independent solution of (2.3.4) *on $I \subseteq \mathbb{R}$ Then every solution y of* (2.3.4) *can be write as a linear combination:*

$$y = c_1y_1 + c_2y_2,$$

where c_1, c_2 are arbitray constants.

Solution of (2.3.4) **:**
We research the solution of the form $y = e^{rx}$, $r \in \mathbb{R}$.
We have $y' = re^{rx}$ and $y'' = r^2e^{rx}$ sunbstituting into (2.3.4) we obtain

$$ar^2 + br + c = 0. \tag{2.3.5}$$

(2.3.5) called the characteristic equation.
We have $\Delta = b^2 - 4ac$, and the following results:

Sign of Δ	The solutions of (2.3.5)	The solutions of (2.3.4)
$\Delta > 0$	$\exists r_1, r_2 \in \mathbb{R}, (r_1 \neq r_2)$	$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
$\Delta = 0$	$\exists r_1 = r_2$	$y = (c_1 + c_2 x) e^{r_1 x}$
$\Delta < 0$	$\exists r_1 = \alpha + i\beta, r_2 = \alpha - i\beta$	$y = \big(c_1 cos(\beta x) + c_2 sin(\beta x)\big) e^{\alpha x}$.

Where c_1, c_2 are arbitrary constants, α and β are constants.

Examples 2.3.3.
Solve the equations:

1. $y^{''} - 3y^{'} + 2y = 0$

2. $y^{''} + 2y^{'} + 2y = 0$

3. $y^{''} - 4y^{'} + 4 = 0.$

Solution:

1) $y^{''} - 3y^{'} + 2y = 0$
The characteristic equation is:

$$r^2 - 3r + 2 = 0.$$

We have $\Delta = (-3)^2 - 4(1)(2) = 1$, there exist two roots $r_1 = 1$ and $r_2 = 2$. Then
The general solution of 1) is: $y = c_1 e^x + c_2 e^{2x}$, where $c_1, c_2 \in \mathbb{R}$.
2)$y^{''} + 2y^{'} + 2y = 0$
The characteristic equation is:

$$r^2 + 2r + 2 = 0.$$

We have $\Delta = (2)^2 - 4(1)(2) = -4 = (2i)^2$, there exist two roots $r_1 = -1 - i$ and $r_2 = -1 + i$, we put $\alpha = -1, \beta = 1$. Then
The general solution of 2) is: $y = (c_1 cosx + +c_2 sinx)e^{-x}$, where $c_1, c_2 \in \mathbb{R}$.
3) For the equation 3) we obtain, the general solution is: $y = (c_1 + c_2 x)e^{2x}$.

2.3.3 Solution of the nonhomogeneous equation

Let the nonhomogeneous equation with constant coefficients

$$ay^{''} + by^{'} + cy = f(x). \tag{2.3.6}$$

The associated characteristic equation is:

$$ar^2 + br + c = 0, a \neq 0. \tag{2.3.7}$$

We know that the general solution of (2.3.6) is: $y = y_h + y_p$.
We choose a particular solution y_p of (2.3.6) as the following table.

Type of $f(x)$	Type of y_p
$f(x) = p_n(x)e^{\lambda x}$, $(\lambda \in \mathbb{R})$	1) If λ is not root of (2.3.7): $y_p = q_n(x)e^{\lambda x}$. 2) If λ is simple root of (2.3.7): $y_p = xq_n(x)e^{\lambda x}$. 3) If λ is double root of (2.3.7): $y_p = x^2 q_n e^{\lambda x}$.
$f(x) = p_n(x)sin(\omega x) + p_m(x)cos(\omega x)$	1) If $i\omega$ is not root of (2.3.7): $y_p = q_n(x)sin(\omega x)$ $+q_m(x)cos(\omega x)$ 2) If $i\omega$ is root of (2.3.7): $y_p = x[q_n(x)sin(\omega x)$ $+q_m(x)cos(\omega x)]$.

Where $p_n(x)$ and $q_m(x)$ are polynomials of degree n, m and λ, ω are constants.

Exercise 2.3.1.
Solve the following equations:

1. $y^{''} + 2y^{'} + 2y = 2x$
2. $y^{''} + y = 2x^2 - 1$
3. $y^{''} - 3y^{'} + 2y = e^x, y(0) = 0, y^{'}(0) = 1$

Solution:

1) $y^{''} + 2y^{'} + 2y = 2x$.
The associated homogeneous is: $y^{''} + 2y^{'} + 2y = 0$, has solution is: $y_h = (c_1 cosx + +c_2 sinx)e^{-x}$, where $c_1, c_2 \in \mathbb{R}$. (From examples (2.3.3)).
We give a particular solution of 1):
We put $f(x) = 2xe^{0x}$, since $\lambda = 0$ is root not of the characteristic equation, then y_p is in the form: $y_p = ax + b$.
We have: $y_p^{'} = a, y_p^{''} = 0$, we substituting in 1) we obtain:
$2a + 2ax + 2b = 2x$, so: **a=1** and **b = -1**, therefore: $y_p = x - 1$, then the general solution of 1) is:
$y = (c_1 cosx + +c_2 sinx)e^{-x} + x - 1$.

2) For $y^{''} + y = 2x^2 - 1$, the general solution is:
$y = c_1 cosx + c_2 sinx + 2x^2 - 5$.

3) $y^{''} - 3y^{'} + 2y = e^x$.
The associated homogeneous is: $y^{''} - 3y^{'} + 2y = 0$, has solution is: $y = c_1 e^x + c_2 e^{2x}$, where $c_1, c_2 \in \mathbb{R}$. (From examples (2.3.3)).
We search a particular solution of 3):
We suppose that $f(x) = e^x$, since $\lambda = 1$ is root of the characteristic equation, then y_p is in the form: $y_p = axe^x$.
We have: $y^{'} = (ax + a)e^x, y^{''} = (ax + 2a)e^x$, we substituting in 3) we get:

$a=-1$, therefore $y_p=-xe^x$, then the general solution of 3) is:
$y=(c_1-x)e^x+c_2e^{2x}$.
We have: $y^{'}=-e^x+(c_1-x)e^x+2c_2e^{2x}$

$$\begin{cases} y(0)=0 \\ y^{'}(0)=1 \end{cases} \Longleftrightarrow \begin{cases} c_1=-c_2 \\ -1+c_1+2c_2=1 \end{cases} \Longleftrightarrow \begin{cases} c_1=-2 \\ c_2=2. \end{cases}$$

So, the solution of 3) is: $y=(-2-x)e^x+2e^{2x}$

Chapter 3

Matrices

3.1 Definitions and examples

Definition 3.1.1.
A rectangular arrangement of m rows and n columns and enclosed within a bracket is called a matrix. We shall denote matrices by capital letters as A, B, C etc.

$$A = \begin{pmatrix} a_{11} & a_{12} & ...a_{1n} \\ a_{21} & a_{22} & ...a_{2n} \\ . & . & . \\ . & . & . \\ . & . & . \\ a_{m1} & a_{m2} & ...a_{mn} \end{pmatrix} = (a_{ij})_{m\times n},$$

where $1 \leq i \leq m$, and $1 \leq j \leq n$. A is a matrix of order $m \times n$.

Examples 3.1.1.
1) $M_1 = \begin{pmatrix} 4 & 1 \\ 0 & -5 \\ -2 & 7 \end{pmatrix}$ *is a matrix of order* 3×2.
2) $M_2 = \begin{pmatrix} i & 3 \\ -3 & 1 \end{pmatrix}$ *is a matrix of order* 2×2*, with coefficients in* $\mathbb{C}$.

Remark 3.1.1.
The set of the matrix of order $m \times n$ denoted by: $\mathrm{M}(\mathbb{R})$ *or* $\mathrm{M}(\mathbb{C})$.

3.2 Matrix Operations [1]

Let $A = (a_{ij})$ and $B = (b_{ij}) \in M_{m,n}(\mathbb{C}), C = (c_{ij}) \in M_{n,p}(\mathbb{C})$ matrices, and $\alpha \in \mathbb{R}$.

1. **Equality of two matrices:**$A = B \iff a_{ij} = b_{ij}, \forall 1 \leq i \leq m, 1 \leq i \leq n.$

   Example 3.2.1.
   $\begin{pmatrix} 1 & 0 \\ 5 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 5 & 3i^2 \end{pmatrix}$, *because* $i^2 = -1$.
   But $\begin{pmatrix} 1 & 3 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

2. **Addition:**
   The sum of A and B, denoted $A + B$, is defined to be the matrix $C = [c_{ij}]$, with: $c_{ij} = a_{ij} + b_{ij}$.

3. **Scalar Multiplication:**
   The product of $\alpha \in \mathbb{C}$ with A, denoted αA, where $\alpha A = (\alpha a_{ij}) = A\alpha$

   Examples 3.2.1.

   (a) $\begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 6 & -3 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 7 & -5 \\ 4 & 4 \end{pmatrix}$

   (b) $2. \begin{pmatrix} 3 & -1 & 2i \\ -2 & 0 & 1 \\ 5 & i & 4 \end{pmatrix} = \begin{pmatrix} 6 & -2 & 4i \\ -4 & 0 & 2 \\ 10 & 2i & 8 \end{pmatrix}$

4. **Matrix Multiplication:**
   The product of A and C, denoted AC, is a matrix $AC = \Sigma_{k=1}^{k=n}(a_{ik}c_{kj}) = a_{i1}c_{1j} + a_{i2}c_{2j} + ... + a_{in}c_{nj}$

   Example 3.2.2.
   $\begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 3 \\ -1 & -4 & 1 \end{pmatrix} = \begin{pmatrix} 1.2 + (-2).(-1) & 8 & 1 \\ 6 & 0 & 9 \end{pmatrix} = \begin{pmatrix} 4 & 8 & 1 \\ 6 & 0 & 9 \end{pmatrix}$

Remarks 3.2.1.

1. *AB is defined if and only if the number of columns of A equal the number of rows of B.*

2. *$A.B \neq B.A$*
   For example: $\begin{pmatrix} 2 \\ -3 \end{pmatrix} \begin{pmatrix} 1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 8 \\ -3 & -12 \end{pmatrix}$.
   But the product $\begin{pmatrix} 1 & 4 \end{pmatrix} . \begin{pmatrix} 2 \\ -3 \end{pmatrix} = (-10)$.

3. *$A.C = B.C$ is not deduce $A = B$.*
   For example: $\begin{pmatrix} 1 & 4 \end{pmatrix} . \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 & 4 \end{pmatrix} . \begin{pmatrix} -2 \\ -2 \end{pmatrix} = (-10)$, *but* $\begin{pmatrix} 2 \\ -3 \end{pmatrix} \neq \begin{pmatrix} -2 \\ -2 \end{pmatrix}$.

3.2.1 Special Matrices

Definition 3.2.1.
Let $A = (a_{ij}) \in M_{(m,n)}(\mathbb{C})$

1. ***Zero-matrix:***
   A is called a zero-matrix, denoted $0_{(m,n)}$ *iff:* $a_{ij} = 0, \forall i, j$ *where* $1 \leq i \leq m, 1 \leq j \leq n$.
   For example: $o_{(2,3)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
2. ***Square matrix:***
   A is called a ***square*** *matrix Iff* $m = n$, *and denoted by:* $A \in M_n(\mathbb{C})$.

3. *Let* $A \in M_n(\mathbb{C})$.
   (a) Diagonal matrix:
   A is said to be a diagonal matrix iff: $\forall i \neq j : a_{ij} = 0$, *denoted* $A = diag(a_{11}, ..., a_{nn})$.
   Examples:
   The zero matrix 0_n *and* $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ *are diagonal matrices*
   (b) Identity matrix:
   A diagonal matrix A is said to be a identity matrix iff: $a_{ii} = 1, \forall 1 \leq i \leq n$, *and denoted by* I_n.
   Examples: $I_1 = 1, I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
   (c) Upper triangular:
   A is said to be an ***upper triangular*** *matrix iff:* $a_{ij} = 0, \forall i > j$.
   (d) lower triangular:
   A is said to be a ***lower triangular*** *matrix iff:* $a_{ij} = 0, \forall i < j$.
   Examples:

   a) $A = \begin{pmatrix} 5 & 3 \\ 0 & -4 \end{pmatrix}$ *is upper triangular and* $B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 7 & 2 & 2i \end{pmatrix}$
   is lower triangular.

b) *The matrices $0_n, I_n$ are upper as well as lower triangular matrices.*

Remark 3.2.1.
A matrix A is a diagonal matrix iff: is upper triangular and lower triangular.

3.2.2 Transpose

Definition 3.2.2.
The transpose of $A = (aij) \in M_{(m,n)}(\mathbb{C})$, denoted A^T is the nm matrix whose ***columns*** *are the respective* ***rows*** *of A, i.e $A^T = (a_{ji})_{n\times m}$.*

Examples 3.2.2.

- $\begin{pmatrix} 1 & 0 \\ 2 & 7 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 \\ 0 & 7 \end{pmatrix}$
- $\begin{pmatrix} 1 & 0 & 3i \\ 3 & -4 & -1 \\ 6 & 0 & 2 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 & 6 \\ 0 & -4 & 0 \\ 3i & -1 & 2 \end{pmatrix}$

Properties 3.2.1.
Let $A \in \mathbb{M}_{(m,n)}(\mathbb{C})$ and $B \in \mathbb{M}_{(n,p)}(\mathbb{C})$, $\lambda \in \mathbb{R}$ (or $\mathbb{C}$).

1. $(A^T)^T = A$
2. $(AB)^T = B^T A^T$
3. $(\lambda A)^T = \lambda A^T$

Definition 3.2.3.
The square matrix $A = (a_{ij}) \in \mathbb{M}_{(m,n)}(\mathbb{C})$ is symmetric iff: $A^T = A$, i.e $a_{ij} = a_{ji}, \forall i, j \in [1, n]$.
A is skew symmetric (or antisymmetric) iff: $A^T = -A$, i.e $a_{ij} = -a_{ji}, \forall i, j \in [1, n]$.

Examples 3.2.3.

- *The matrixs $A = \lambda$ and $B = \begin{pmatrix} 1 & -2 & 5 \\ -2 & 3 & 0 \\ 5 & 0 & 7 \end{pmatrix}$ are symmetric because $A^T = A$, and $B^T = B$, where $\lambda \in \mathbb{C}$.*

- *The matrix* $C = \begin{pmatrix} 4 & -2 \\ -2 & 6 \end{pmatrix}$ *is skew symmetric.*
- *The matrix* $D = \begin{pmatrix} 1 & -2 & 5 \\ 2 & 3 & 0 \\ 5 & 0 & 7 \end{pmatrix}$ *is neither symmetric nor antisymmetric, because* $D^T \neq D$ *and* $D^T \neq -D$.

Exercise 3.2.1. *Let the matrix* $A = \begin{pmatrix} 1 & a & b \\ -4 & 5 & 8 \\ 0 & c & 7 \end{pmatrix}$
Find the $a, b \in \mathbb{R}$, *such that:*

- A *is symmetric.*
- A *skew symmetric.*

Exercise 3.2.2.
Calculate $x \in \mathbb{R}$, *such that:*

$$\begin{pmatrix} x & 1 \\ 3 & -2 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ 9 & 8 \\ 20 & 8 \end{pmatrix}$$

3.3 The Determinant

The determinant of the square matrix A the order n is the number denoted by $det(A)$ or $|A|$.
1) If $n = 1$: i.e $A = [a]$,then $det(A) = a$
2) If $n = 2$: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $det(A) = ad - cb$.

Example 3.3.1.
Let $A = \begin{pmatrix} 4 & 2 \\ 5 & -1 \end{pmatrix}$, *then* $det(A) = 4(-1) - (5)(2) = -14$.

Definition 3.3.1.
Let A *a square matrix of order* n.*Then the determinant of* A *is defined by:*
1) If $n = 1$: $A = [a], det(A) = a$.
2) If $n \neq 1$: $det(A) = \sum_{i=1}^{i=n} (-1)^{i+j} a_{ij} det(A_{ij}) = \sum_{j=1}^{i=n} (-1)^{i+j} a_{ij} det(A_{ij})$, *where* A_{ij} *is a matrix minor of order* $(n-1)$, *formed by deleting the* i^{th} ***row*** *and the* j^{th} ***column*** *of* A *(we can be fixed the* ***row*** i *or the* ***column*** j*).*

Particular case n=3:

Let $A = \begin{pmatrix} a_{11}^{+} & a_{12}^{-} & a_{13}^{+} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then

$det(A) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Examples 3.3.1.

Let $A = \begin{pmatrix} 1 & -2 \\ 4 & 3 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 1 \\ -1 & 2 & -2 \end{pmatrix}$.

Find the matrix minors A_{12}, B_{11}, B_{21}.

Solution:

A_{12} is a matrix minor of order 2, formed by deleting the **first row** and the **second column** of A. Then

$A_{12} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$

B_{11} is a matrix minor of order 2, formed by deleting the **first row** and the **first column** of B. Then

$B_{11} = \begin{pmatrix} 3 & 1 \\ 2 & -2 \end{pmatrix}$, and $B_{21} = \begin{pmatrix} 2 & 3 \\ 2 & -2 \end{pmatrix}$

Example 3.3.2.

Let $A = \begin{pmatrix} 1^{+} & 2^{-} & 3^{+} \\ 0 & 3 & 1 \\ -1 & 2 & -2 \end{pmatrix}$

We have:

$$\begin{aligned} det(A) &= 1.det(A_{11}) - 2.det(A_{12}) + 3.det(A_{13}) \\ &= 1.\begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix} - 2.\begin{vmatrix} 0 & 1 \\ -1 & -2 \end{vmatrix} + 3.\begin{vmatrix} 0 & 3 \\ -1 & 2 \end{vmatrix} \\ &= -8 - 2 + 9 = -1 \end{aligned}$$

Remark 3.3.1.

We can calculate $det(A)$ *by:* $det(A) = 1.det(A_{11}) - 0.det(A_{21}) - 1.det(A_{31})$

Example 3.3.3.

Let $B = \begin{pmatrix} 5 & -1 & 3 \\ 0 & 1 & 2 \\ 4 & -2 & 6 \end{pmatrix}$, *then*

$$\begin{aligned} det(B) &= 5.det(B_{11}) - 0.det(B_{21}) + 4.det(B_{31}) \\ &= 5.\begin{vmatrix} 1 & 2 \\ -2 & 6 \end{vmatrix} + 4.\begin{vmatrix} -1 & 3 \\ 1 & 2 \end{vmatrix} \\ &= 30 - 20 = 10 \end{aligned}$$

Properties 3.3.1.
Let $A = (a_{ij})$ and B be two square matrices of the same order n, and I_n is a matrix identity. Then

1. *If A is a triangular matrix: $det(A) = \prod_{i=1}^{n} a_{ii} = a_{11}a_{22}...a_{nn}$.*
   In particular: $det(I_n) = 1$

2. $det(A^T) = det(A)$

3. $det(\lambda A) = \lambda^n det(A)$, *where* $\lambda \in \mathbb{C}$.

4. $det(AB) = det(A).det(B)$.

Examples 3.3.2.

1. *Let* $A = \begin{pmatrix} 2 & 4 & -12 \\ 0 & -4 & 6 \\ 0 & 2 & 10 \end{pmatrix} = 2.\begin{pmatrix} 1 & 2 & -6 \\ 0 & -2 & 3 \\ 0 & 1 & 5 \end{pmatrix}$, *then*
   $det(A) = 2^3.\begin{vmatrix} 1 & 2 & -6 \\ 0 & -2 & 3 \\ 0 & 1 & 5 \end{vmatrix} = 8.\begin{vmatrix} -2 & 3 \\ 1 & 5 \end{vmatrix} = 8.(-13) = -64$

2. *Let the triangular matrix* $B = \begin{pmatrix} 1 & 7 & 10 & -5 \\ 0 & 2 & -8 & 9 \\ 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 5 \end{pmatrix}$, *then*
   $det(B) = (1).(2).(-4).(5) = -40$.

Theorem 3.3.1. ***(Special matrices with Zero determinant)***

Let A be an $n \times n$-matrix.
1. *If A has **a row** consisting only of **zeros**, or **a column** consisting only of **zeros**, then $det(A) = 0$.*
2. *If A has **a row** that is **a scalar multiple** of **another row**, or **a column** that is **a scalar multiple** of **another column**, then $det(A) = 0$.*

Examples 3.3.3.

1. *Let* $A = \begin{pmatrix} 0 & 8 & -10 \\ 0 & -4 & i \\ 0 & 3 & 1 \end{pmatrix}$*, then* $det(A) = 0$.

2. *Let* $B = \begin{pmatrix} 2 & 4 & -6 \\ 1 & 2 & -3 \\ 7 & 0 & 5 \end{pmatrix}$*, then* $det(B) = 0$*, because the first row is multiple two of the second row.*

3.4 The inverse of a matrix

Definition 3.4.1.
Let $A \in \mathbb{M}_n(\mathbb{C})$ A *is invertible (has an inverse) if only if there exists* $B \in \mathbb{M}_n(\mathbb{C})$ *such that* $AB = BA = In$*, we denoted by* $B = A^{-1}$.

Example 3.4.1.
Let $P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ *and* $Q = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$, $P = Q^{-1}$ *because* $PQ = QP = I_2$.

3.4.1 The cofactor matrix

Definition 3.4.2.
Let $A \in M_n(\mathbb{C})$*. Then, the cofactor matrix, denoted* $Cof(A)$*, is an* $M_n(\mathbb{C})$ *matrix with* $Cof(A) = [Cij]$*, where*

$$C_{ij} = (-1)^{i+j} det(A_{ij}),\ for \quad 1 \leq i \leq n, 1 \leq j \leq n.$$

And, the Adjugate (classical Adjoint) of A*, denoted* $Adj(A) = Cof^T(A)$.

Example 3.4.2.
Let $A = \begin{pmatrix} 1^+ & 2^- & 3^+ \\ 2 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$.

$$\text{We have } Cof(A) = \begin{pmatrix} +\begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} \\ +\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 10 & -7 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{pmatrix},$$

$$\textit{and } Adj(A) = Cof^T(A) = \begin{pmatrix} 10 & -2 & -7 \\ -7 & 1 & 5 \\ 1 & 0 & -1 \end{pmatrix}$$

3.4.2 Formula for the inverse

Theorem 3.4.1.
Let A is a square matrix of order n.
A is invertible iff: $det(A) \neq 0$. In this case, we have

$$A^{-1} = \frac{1}{det(A)}.Adj(A) = \frac{1}{det(A)}.Cof^T(A)$$

Examples 3.4.1.
Find the inverse for all matrix if exist
a) $A = \begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix}$
b) $B = \begin{pmatrix} 3 & -6 \\ 1 & -2 \end{pmatrix}$
c) $C = \begin{pmatrix} 0 & 2 & 1 \\ -1 & 2 & 2 \\ 2 & -2 & -2 \end{pmatrix}$

<u>Solution</u>

a) We have $det(A) = 12 - 04 = 2 \neq 0$, then A is invertible and
$A^{-1} = \frac{1}{det(A)}.Cof^T(A)$, where $Cof(A) = \begin{pmatrix} 2 & 0 \\ -4 & 1 \end{pmatrix}$, so
$A^{-1} = \frac{1}{2}\begin{pmatrix} 2 & -4 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & \frac{1}{2} \end{pmatrix}$.
b) We have $det(B) = 3(-2) - 1(-6) = 0$, then B is not invertible.
c) We have $det(C) = 0.\begin{vmatrix} 2 & -2 \\ -2 & -2 \end{vmatrix} - 2.\begin{vmatrix} -1 & 2 \\ 2 & -2 \end{vmatrix} + 1.\begin{vmatrix} -1 & 2 \\ 2 & -2 \end{vmatrix}$= -2(-2)-2= 2.

Since $det(C) \neq 0$, then C is invertible and $C^{-1} = \frac{1}{det(C)}.Cof^T(C)$, where

$$Cof(C) = \begin{bmatrix} +\begin{vmatrix} 2 & 2 \\ -2 & -2 \end{vmatrix} & -\begin{vmatrix} -1 & 2 \\ 2 & -2 \end{vmatrix} & +\begin{vmatrix} -1 & 2 \\ 2 & -2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ -2 & -2 \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ 2 & -2 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} \\ +\begin{vmatrix} 0 & 2 \\ 2 & -2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} \end{bmatrix} = \begin{pmatrix} 0 & 2 & -2 \\ 2 & -2 & 4 \\ 2 & -1 & 2 \end{pmatrix}.$$

Therefore

$C^{-1} = \frac{1}{2}\begin{pmatrix} 0 & 2 & 2 \\ 2 & -2 & -1 \\ -2 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & -\frac{1}{2} \\ -1 & 2 & 1 \end{pmatrix}.$

Exercise 3.4.1.
Determine whether each of the following matrices is invertible? If yes, find the inverse.
$A = \begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 4 & 1 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 1 & 2 \\ -2 & 2 & 4 \end{pmatrix}$.

Exercise 3.4.2.
Consider the matrix

$$\begin{pmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ t & 0 & 2 \end{pmatrix}, t \in \mathbb{R}.$$

Does there exist a value of t for which this matrix fails to be invertible? Explain.

3.5 Linear application associated of the matrix

Definition 3.5.1.
We say that the application $f : E \longrightarrow F$ is linear iff:

$$\forall (X, Y) \in E^2, \forall (\alpha, \beta) \in \mathbb{R}^2 : f(\alpha X + \beta Y) = \alpha f(X) + \beta f(Y)$$

Examples 3.5.1.

1. $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = 4x$ *is linear because:*
   $\forall (x, y) \in \mathbb{R}^2, \forall (\alpha, \beta) \in \mathbb{R}^2 : f(\alpha x + \beta y) = 4(\alpha x + \beta y) = \alpha(4x) + \beta(4y) = \alpha f(x) + \beta f(y)$.

2. $g : \mathbb{R} \longrightarrow \mathbb{R}, g(x) = 3x + 1$ *is not linear, because* $g(\alpha x + \beta y) = 3(\alpha x + \beta y) + 1 = 3\alpha x + 3\beta y + 1 \neq \alpha g(x) + \beta g(y)$.

Definition 3.5.2.
We call the linear application associated of the matrix $P = (a_{ij}) \in \mathcal{M}_{m,n}$, the application f defined by:

$$f : R^n \longrightarrow R^m,$$

where $\forall X \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix} \in R^n, f(X) = P.X$

Example 3.5.1.
The linear application associated of the matrix $A = \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$ *is*
$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$\forall X \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2,\ f(X) = A.X = \begin{pmatrix} x + 2y \\ -3x + 4y \end{pmatrix}.$

Proposition 3.5.1.
The application

$$f : R^n \longrightarrow R^m,\ where f(X) = P.X,$$

is bijective iff: P is invertible i.e $det(P) \neq 0$, in this case :

$$f^{-1} : R^m \longrightarrow R^n,$$

where $f^{-1}(X) = P^{-1}.X$.

Exercise 3.5.1.
Let $A = \begin{pmatrix} 3 & 5 \\ -1 & -2 \end{pmatrix}$

1. *Prove that A is invertible and find A^{-1}.*

2. *Determine the application associated of the matrix A.*

3. *Find the inverse application f^{-1}.*

Solution:

1) *We have* $det(A) = 3(-2) + 5 = -1 \neq 0$, *then A is invertible.*
$A^{-1} = \frac{1}{det(A)}.Cof^T(A)$, *where*
$Cof(A) = \begin{pmatrix} -2 & 1 \\ -5 & 3 \end{pmatrix}$, *so:* $A^{-1} = \begin{pmatrix} 2 & -1 \\ 5 & -3 \end{pmatrix}$.

2) *The application associated of A is:*
$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$,*such that*
$\forall X \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2, f(X) = A.X = \begin{pmatrix} -2x + y \\ -5x + 3y \end{pmatrix}.$

3) *The inverse application is:* $f^{-1} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$,*such that*
$\forall X \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2, f(X) = A^{-1}.X = \begin{pmatrix} 2x - y \\ 5x - 3y \end{pmatrix}.$

3.6 Change of Basis, Transition Matrix

Definition 3.6.1. ***(Vector Space)***
We said that the non-empty set E is a vector space over K (where $K = \mathbb{R}$ or $K = \mathbb{C}$), or K-vector space if E is equipped with two composition laws:
for all $\alpha, \beta \in \mathbb{R}$ and all $u, v, w \in E$
Internal composition law *for "addition" verifying:*

1. $u + v = v + u$ *(commutative law for addition)*

2. $u + (v + w) = (u + v) + w$ *(associative law for addition)*

3. $\exists 0_E \in E$, *such that* $\forall u \in E, 0_E + u = u$ *(0_E called neutral element).*

4. $\forall u \in E, \exists v \in E, u + v = 0_E$, *($v = -u$ called symmetric element).*

External composition law *"scalar multiplication" verifying:*

1. $\alpha(u + v) = \alpha u + \alpha v$

2. $(\alpha + \beta)u = \alpha u + \beta u$

3. $\alpha(\beta x) = (\alpha\beta)x$

4. $\exists 1_E \in E$, *such that,* $\forall u \in E : 1_E u = u$.

The elements of E are called to as "vectors."

Examples 3.6.1.

- $(\mathbb{R}^2, +, .)$ *is vector space, where* $+, .$ *defined by:* $\forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, $\forall \alpha \in \mathbb{R}$:
  $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$,
  and $\alpha.(x_1, y_1) = (\alpha x_1, \alpha y_1)$.

- $(\mathbb{R}^n, +, .)$ *is vector space.*

- *The set of the matrices* $\mathcal{M}_{2,2}(\mathbb{R}),+,.)$ *is vector space*

Definition 3.6.2. ***(Linear Independence)***

. *A family* $(x_1, x_2, ..., x_n)$ *of* E *is called linearly independent if and only if:* $\forall \alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{K}$*:*

$$\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n = 0 \Longrightarrow \alpha_1 = \alpha_2 = ... = \alpha_n = 0.$$

.. *A family* $(x_1, x_2, ..., x_n)$ *of* E *is said to be generating if for every vector* $X \in E$*, can be exist* $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{K}$*, such that:*

$$X = \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n$$

Definition 3.6.3.
A vector space's basis is defined as any family $(x_1, x_2, ..., x_n)$ *of* E *that is both linearly independent and generating for* E*.*

Example 3.6.1.
$B = \{(1,0),(0,1)\}$ *is a basis of* $\mathbb{R}^2$*,because:*
a) B *linearly independent: We have* $\forall(\alpha, \beta) \in \mathbb{R}^2$*,* $\alpha(1,0) + \beta(0,1) = 0 \Longrightarrow \alpha = \beta = 0$*.*
b) B *is generating of* $\mathbb{R}^2$ *:* $\forall(x,y) \in \mathbb{R}^2$*, we have* $(x,y) = x(1,0) + y(0,1)$*.*

3.6.1 Change of basis matrix

In a vector space E of dimension n, consider two bases:
$B_1 = (e_1, e_2, ..., e_n)$, $\acute{B} = (\acute{e_1}, \acute{e_2}, \acute{e_n})$.
If the vectors $\acute{e}_j$ are defined in the basis B by the formula:

$$\acute{e}_j = \sum_{i=1}^{n} \alpha_{ij} e_i,$$

then, the change of basis matrix from the basis B to the basis B0 is given by:

$$P = (\alpha_{ij}) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & . & . & . & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & . & . & . & \alpha_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \alpha_{n1} & \alpha_{n2} & . & . & . & \alpha_{nn} \end{pmatrix}.$$

Therefore if $X = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_1 \end{pmatrix}$, $Y = \begin{pmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_1 \end{pmatrix}$, the two matrices of a vector u in B and $\acute{B}$ we have the following results:

$$X = P.Y \ \ or \ \ Y = P^{-1}.X$$

Chapter 4

Systems of linear equations

4.1 Definition and examples

Definition 4.1.1.
A system of m linear equations in n variables $x_1, x_2, \cdots, x_n$ is a set of equations of the form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ . \\ . \\ . \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_n \end{cases} \tag{4.1.1}$$

where $a_{ij},\ b_i \in \mathbb{R}\ (1 \leq i \leq m,\ 1 \leq j \leq n)$.

Remarks 4.1.1.

1. (4.1.1) *is called homogeneous if $b_i = 0$, $\forall 1 \leq i \leq n$ and **non-homogeneous**, otherwise.*

2. *Let* $A = \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdot & \cdot & \cdot & a_{mn} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$, $B = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{pmatrix}$

   (4.1.1) *can be we written as $AX = B \Leftrightarrow$* (4.1.1), *where A is called the **coefficient matrix**.*

Examples 4.1.1.

1. *Let the system*

$$\begin{cases} 2x - 3y = 4 \\ y + 5y = 1 \end{cases} \tag{4.1.2}$$

The associated matrix is $A = \begin{pmatrix} 2 & -3 \\ 1 & 5 \end{pmatrix}$

$(4.1.2) \Leftrightarrow A \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

2. *Let the system*

$$\begin{cases} x - 5y = 4 \\ 3x - 2y + z = 1 \\ y - 2z = 6 \end{cases}$$

The matriciel form is: $B.X = C$, *where* $B = \begin{pmatrix} 1 & -5 & 0 \\ 3 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$ *and* $C = \begin{pmatrix} 4 \\ 1 \\ 6 \end{pmatrix}$.

4.2 Rank of a matrix

Definition 4.2.1.
Consider $A = (a_{ij}) \in \mathcal{M}_{m,n}(\mathbb{C})$

1. $Rank(A) = r \geq 1$ *iff* A *has a* $(r \times r)$ *sub matrix with* ***nonzero*** *determinant.*

2. *An* $(n \times n)$ *square matrix* A *has* $Rank(A) = n$ *iff:*

$$det(A) \neq 0$$

Examples 4.2.1.

1. $A = \begin{pmatrix} 5 & 15 \\ 1 & 3 \end{pmatrix} \in \mathcal{M}_{2\times 2}$
   We have: $det(A) = 0$ *Then:* $A_1 = [5]$, $det(A_1) \neq 0$ $Rank(A) = 1$

2. $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 1 \\ 2 & -2 & -1 \end{pmatrix}$
   We have: $det(A) = 0$, $A_{3,3} = A$, *then* $Rank(A) \neq 3$.
   $A_{2,2} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$, $det(A_{2,2}) \neq 0$. *Then:* $Rank(A) = 2$.

4.3 The set solution of system $AX = B$ (4.1.1)

We put $Rank(A) = r$, $A \in \mathcal{M}_{m,n}(\mathbb{C})$.

1. **If** $r = m = n$ **and** $det(A) \neq 0$: The system 4.1.1 has a unique solution
2. **If** $r < n$ $(det(A) = 0)$: The system $AX = B$ has an infinite solution or not exist a solution.

4.4 Method of solution

4.4.1 Cramer's Rule

Let $AX = B$, $det(A) \neq 0$. We have the theorem.

Theorem 4.4.1. *(Cramer's Rule)*
Suppose $det(A) \neq 0$ $(A \in \mathcal{M}_{n,n})$ *and we wish to solve the system* $AX = B$ *by:*

$$x_i = \frac{det(A_i)}{det(A)},$$

where: A_i *is the matrix obtained by replacing the* i *th column of* B.

Example 4.4.1.
Let the system

$$\begin{cases} 3x - y = 4 \\ -5y + 2y = -2 \end{cases} \tag{4.4.1}$$

$$A = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix},\ B = \begin{pmatrix} 4 \\ -2 \end{pmatrix},\ X = \begin{pmatrix} x \\ y \end{pmatrix}$$

(4.4.1) $\Leftrightarrow$ $AX = B$

$det(A) = 3.2 - 5 = 1 \neq 0$. *Then:* (4.4.1) *has a unique solution.*

$$x = \frac{det(A_1)}{det(A)}, \quad y = \frac{det(A_2)}{det(A)},$$

$$A_1 = \begin{pmatrix} 4 & -1 \\ -2 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 3 & 4 \\ -5 & -2 \end{pmatrix}$$

$x = \frac{det(A_1)}{det(A)} = \frac{6}{1} = 6$, $y = \frac{det(A_2)}{det(A)} = \frac{14}{1} = 14$

Example 4.4.2.
Let the system

$$\begin{cases} 2x + y = 5 \\ 6x + 3y = 2 \end{cases} \tag{4.4.2}$$

We consider

$$A = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}, \; B = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \; X = \begin{pmatrix} x \\ y \end{pmatrix}$$

(4.4.2) $\Leftrightarrow \; AX = B$

$det(A) = 2 \times 3 - 6 \times 1 = 0$. *Then:* (4.4.2) *has an infinite solution or not exist a solution.*
We have: $6x + 3y = 2 \;\Leftrightarrow\; 2x + y = 1$, *then*
(4.4.2) $\Leftrightarrow \begin{cases} 2x + y = 5 \\ 2x + y = 1 \end{cases}$
from the two equation we have a contradiction, then the set solution is $S = \phi$.

4.4.2 Method of inverssion

Consider the matrix equation: $AX = B$, $|A| \neq 0$.

$$\begin{aligned} AX = B &\Leftrightarrow A^{-1}AX = A^{-1}B \\ &\Leftrightarrow X = A^{-1}B \end{aligned}$$

Then: the system (4.1.1) has a unique solution X.

Example 4.4.3.

$$\begin{cases} 2x + y = 7 \\ -3y + z = -8 \\ y + 2z = -3 \end{cases}$$

Let: $A = \begin{pmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} 7 \\ -8 \\ -3 \end{pmatrix}$

$det(A) = 4 \neq 0$. *Then:* A *is invertible*

$AX = B \Leftrightarrow X = A^{-1} \cdot B \; (*)$

We have $Com(A) = \begin{pmatrix} -1 & 6 & -3 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{pmatrix}$

Then:

$$A^{-1} = \frac{1}{det(A)} \cdot Com(A)^T = \frac{1}{4}\begin{pmatrix} -1 & -2 & 1 \\ 6 & 4 & -2 \\ -3 & -2 & 3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\ \frac{3}{2} & 1 & -\frac{1}{2} \\ -\frac{3}{4} & -\frac{1}{2} & \frac{3}{4} \end{pmatrix}$$

From $(*), X = A^{-1}B = \begin{pmatrix} -\frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\ \frac{3}{2} & 1 & -\frac{1}{2} \\ -\frac{3}{4} & -\frac{1}{2} & \frac{3}{4} \end{pmatrix}\begin{pmatrix} 7 \\ -8 \\ -3 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ 4 \\ -\frac{7}{2} \end{pmatrix}$

4.4.3 Method of Gaussien

Elementary operations

Let the system of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = & b_2 \\ . & . \\ . & . \\ . & . \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = & b_m \end{cases} \tag{4.4.3}$$

Definition 4.4.1.
The Augmented matrix of (4.4.3) ***is*** $[A|B]$ *there are six types of elementary transformations of A, three of them are row transformations and other three of them are column transformations.*
*There are as follows "**Elementary row operations***

1. Interchange two rows denoted by: $R_i \leftrightarrow R_j$.
2. Multiplication by $K \in \mathbb{R}^*$ to all elements in the i^{th} row denoted by:

$$R_i \longleftarrow KR_j$$

3. Add a multiple of one row to another row denoted by:

$$R_i \longleftarrow R_i + KR_j$$

Definition 4.4.2. ***(Equivalent matrix)***
A matrix B is said to be equivalent to a matrix A if B can be obtained from A, by many successive elementary transformations on a matrix A.

$$A \sim B.$$

Definition 4.4.3. ***Echelon from (upper triangular)***
A system of three equations in variables x, y, z is said to be in echelon from iff: can be written:

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ b_2y + c_2z = d_2 \\ c_3z = d_3 \end{cases}$$

i.e: $[A|B] = \left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2 & c_2 & d_2 \\ 0 & 0 & c_3 & d_3 \end{array}\right)$

To solve a system $AX = B$ if can be transformed $[A|B]$ in upper triangular this method called Gaussien's method.

Example 4.4.4.
Solve the system $\begin{cases} 2x - 3y = -10 \\ x - 3y = -8 \end{cases}$

Solution:

The augmented matrix is:

$$\left[\begin{array}{cc|c} 2 & -3 & -10 \\ 1 & -3 & -8 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & -3 & -8 \\ 2 & -3 & -10 \end{array}\right]$$

$$\xrightarrow{R_2 \longleftarrow R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & -3 & -8 \\ 0 & 3 & 6 \end{array}\right]$$

The system corresponding of this from is:

$$\begin{cases} x - 3y = -8 \\ 3y = 6 \end{cases} \Leftrightarrow \begin{cases} y = 2 \\ x = 3 \times 2 - 8 = -2, \end{cases}$$

then the set solutionis: $S = \{(-2, 2)\}$

Example 4.4.5.
Solve the system $\begin{cases} 3x - y + 5z = 8 \\ y - 10z = 1 \\ 6x - y = 17 \end{cases}$

Solution:

The augmented matrix of this system is:

$$[A|B] = \left[\begin{array}{ccc|c} 3 & -1 & 5 & 8 \\ 0 & 1 & -10 & 1 \\ 6 & -1 & 0 & 17 \end{array}\right] \xrightarrow{R_3 \longleftarrow R_3 - 2R_1} \left[\begin{array}{ccc|c} 3 & -1 & 5 & 8 \\ 0 & 1 & -10 & 1 \\ 0 & 1 & -10 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \longleftarrow R_3 - R_2} \left[\begin{array}{ccc|c} 3 & -1 & 5 & 8 \\ 0 & 1 & -10 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system corresponding to this echelon from is:

$$\begin{cases} 3x - y + 5z = 8 \\ y - 10z = 1 \\ 0x + 0y + 0z = 0 \end{cases} \Leftrightarrow \begin{cases} y = 10z + 1 \\ 3x - y + 5z = 8 \\ z = t \end{cases} \Leftrightarrow \begin{cases} z = t \\ y = 10t + 1 \\ x = \frac{1}{3}(10t + 1 - 5t + 8) = 3 + \frac{5}{3}t \end{cases}$$

The set solution is: $S = \left\{(3 + \frac{5}{3}t, 10t + 1, t)\right\}$, $t \in \mathbb{R}$, (infinite solution).

Example 4.4.6.

For this system $\begin{cases} x - 4y + 3z = 11 \\ 2x + 10y + 7z = 27 \\ x + y + 2z = 5 \end{cases}$

The operations are:

$R_2 \leftarrow R_2 - 2R_1,\ R_3 \leftarrow R_3 - R_1$

$R_3 \leftarrow 2R_3,\ R_3 \leftarrow R_3 + 3R_2$

We deduce the set solution is:

$$S = \{(-2, 1, 3)\}$$

Chapter 5

Multivariable functions

5.1 Definitions and examples

Definition 5.1.1.
We call a function of $n (n \in \mathbb{N}^*)$ *real variables any function*

$$\begin{array}{rcl} f : \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ (x_1, \cdots, x_n) & \longmapsto & y \end{array}$$

Examples 5.1.1.

$$\textbf{\textit{1)}}\ \begin{array}{rcl} f : \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & 5x^2 - 1 \end{array}$$

$$\textbf{\textit{2)}}\ \begin{array}{rcl} g : \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x, y) & \longmapsto & (xy - 4) \end{array}$$

Definition 5.1.2.
The definition set of f *are the points of* $M \in \mathbb{R}^n / f(M) \in \mathbb{R}$.

Examples 5.1.2.

- *1)* $f(x, y) = \frac{x}{2y}, \quad D_f = \mathbb{R} \times \mathbb{R}^*$

- *2)* $g(x, y) = \sqrt{x} - \frac{1}{y}, \quad D_g = \mathbb{R}_+ \times \mathbb{R}^*$

5.2 Limits, continuity, and partial derivatives of a function

For simplicity, the statements will be given in the case of two variables

Definition 5.2.1. *(Limits)*
Let $(x_0, y_0) \in D_f$.
$\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = \ell \Leftrightarrow (\forall\varepsilon > 0, \exists\alpha, \beta > 0, |x - x_0| < \alpha \text{ and } |y - y_0| < \beta) \Rightarrow |f(x,y) - \ell| < \varepsilon$

Definition 5.2.2. *(Continuity)*
Let $(x_0, y_0) \in D_f$.

1. *f continues in $(x_0, y_0) \Leftrightarrow \lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = \ell$.*
2. *f continue on $I \in D_f$, iff f continues at all points I*

Examples 5.2.1.

(I) Determine the domain of definition of functions:

$$f(x,y) = \frac{x+y}{x-y}, \qquad D_f = \left\{(x,y) \in \mathbb{R}^2, x \neq y\right\}$$

$$g(x,y) = \frac{x-2y}{x^2+y^2}, \qquad D_g = \mathbb{R}^{*2}$$

$$h(x,y) = \frac{\ln(y)}{\sqrt{x-y}}, \qquad D_h = \left\{(x,y) \in \mathbb{R}^2, y > 0 \text{ and } x > y\right\}$$

(II) Study the continuity of functions:

$$1)\ f(x) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & if \quad (x,y) \neq (0,0) \\ 0 \quad if & (x,y) = (0,0). \end{cases}$$

We have: $D_f = \mathbb{R}$
We pose: $y = 0$, *then*

$$\lim_{(x,0)\to(0,0)} f(x,y) = \lim_{(x,0)\to(0,0)} \frac{x^2}{x^2} = 1$$

We pose: $x = 0$, *then*

$$\lim_{(0,y)\to(0,0)} f(x,y) = \lim_{(0,y)\to(0,0)} \frac{-y^2}{y^2} = -1.$$

So f does not admit a limit. Then f is not continuous at $(0,0)$.

$$2)\ g(x) = \begin{cases} \frac{y^3}{(x-1)^2+y^2} & if \quad (x,y) \neq (1,0) \\ 0 \quad if & (x,y) = (1,0). \end{cases}$$

We have: $D_g = \mathbb{R}^2$

We pose: $x = 1,\ y \neq 0,\ g(1,y) = y$. Then

$$\lim_{(x,y)\to(1,0)} g(x,y) = \lim_{y\to 0} y = 0$$

We pose: $y = 0$ and $x \neq 1$, $g(x,0) = 0$. Then

$$\lim_{(x,y)\to(1,0)} g(x,y) = 0 = g(0,0).$$

So g is continues at a point $(1,0)$.

5.3 Differentiability

Definition 5.3.1.
We say that f is differentiable in (x_0, y_0) iff $\exists \ell_1$ and ℓ_2 such that

$$\ell_1 = \frac{\partial f}{\partial x}(x_0, y_0) \quad and \quad \ell_2 = \frac{\partial f}{\partial y}(x_0, y_0)$$

Remark 5.3.1.
We note: $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ the total differential, and we write

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \\ \frac{\partial f}{\partial y} \end{pmatrix}$$

Example 5.3.1.
Calculate all partial derivatives of order 1 for all functions

1. $f(x,y) = y^2 - 3xy \Rightarrow \frac{\partial f}{\partial x} = -3y$.
2. $g(x,y) = \frac{x-3y}{y+x}$, *and deduce* $\frac{\partial g}{\partial x}(0,1)$.

Solution:
1) We have: $D_f = \mathbb{R}^2$, then $\frac{\partial f}{\partial x} = -3y$ and $\frac{\partial f}{\partial y} = 2y - 3x$.
2)For $(x,y) \in \mathbb{R}^2$ and $y \neq -x$: We have, $\frac{\partial g}{\partial x} = \frac{1.(y+x)-(x-3y)}{(y+x)^2} = \frac{4y}{(y+x)^2}$
$\frac{\partial g}{\partial y} = \frac{-3(y+x)-(x-3y)}{(y+x)^2} = \frac{-2x}{(y+x)^2}$.
We deduce that: $\frac{\partial g}{\partial x}(0,1) = 4$.

5.4 Double and triple integral

5.4.1 Double integral

Theorem 5.4.1.
Let φ and ψ two continuous functions on $[a,b]$ with $\varphi \leq \psi$. We put:

$$D = \left\{(x,y) \in \mathbb{R}^2 / a \leq x \leq b \text{ and } \varphi(x) \leq y \leq \psi(x)\right\}.$$

Then

$$\int\int_D f(x,y)dxdy = \int_a^b \left[\int_{\varphi(x)}^{\psi(x)} f(x,y)dy\right] dx$$

Remarks 5.4.1.

1) Particular case

If: $f(x,y) = 1$, then:

$$\int_D dxdy = \mathcal{A}(D) \quad \text{is the area of } D.$$

2) We can swap the roles of x and y.

Example 5.4.1.
We calculate: $\int\int_\Omega xy^2 dxdy$, *where* $\Omega = [0,1] \times [0,2]$.

$$I = \int_0^1 \left[\int_0^2 xy^2 dy\right] dx = \int_0^1 \left[\frac{1}{3}xy^3 dy\right]_0^2 dx = \int_0^1 \left[\frac{8}{3}x\right] dx = [\frac{4}{3}x^2]_0^1 = \frac{4}{3}.$$

We note that: $I = \int_0^2 \left[\int_0^1 xy^2 dx\right] dy$

Example 5.4.2. *(Circle Area $\mathbb{S}^1 : x^2 + y^2 = 1$).*

We calculate the integral using polar coordinates

$$x = rcos\theta, y = rsin\theta$$

to get

$$\int\int_{\mathbb{S}^1} dxdy = \int_0^{2\pi} \left[\int_0^r rdr\right] d\theta = \int_0^{2\pi} [\frac{1}{2}r^2] d\theta = \pi r^2.$$

5.4.2 Triple integral

$D = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$

$$\int\int\int_D f(x,y,z)dxdydz = \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} \left(\int_{a_3}^{b_3} f(x,y,z)dz \right) dy \right) dx$$

Particular case

If $f = 1$: $\int\int\int_D dxdydz = V(D)$ is volume of D.

Example 5.4.3.

$$\int_0^1 \left[\int_1^2 \left[\int_0^3 (x-y)dz \right] dy \right] dx = \int_0^1 \left[\int_1^2 [(x-y)z]_0^3 \, dy \right] dx$$
$$= \int_0^1 \left[\int_1^2 (x-y)3dy \right] dx$$
$$= \int_0^1 \left[xy - \frac{3}{2}y^2 \right]_1^2 dx$$
$$= \int_0^1 \left(3x - \frac{9}{2} \right) dx$$
$$= \left[\frac{3}{2}x^2 - \frac{9}{2}x \right]_0^1 = \frac{3}{2} - \frac{9}{2} = -3$$

Particular case

$f = 1$ $\int\int\int_D dxdydz = V(D)$ volume of D.

Exercise:

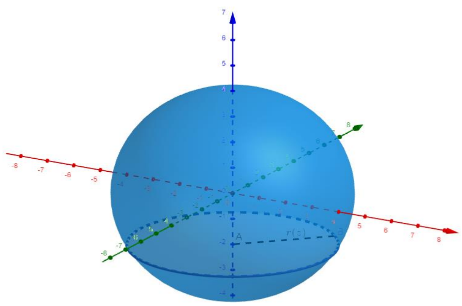
Show that the volume of the sphere $\mathbb{S}^2$ is $V = \frac{4}{3}\pi R^3$, where R is the radius of $\mathbb{S}^2$.

<u>**Solution:**</u>

We have $r^2 = R^2 - z^2$, then

$$V = \int_{-R}^{R} \pi r^2 dz = \int_{-R}^{R} \pi(R^2 - z^2) dz$$

$$= \pi \left[R^2 z - \frac{1}{3} z^3 \right]_{-R}^{R} = \frac{4}{3}\pi R^3$$



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